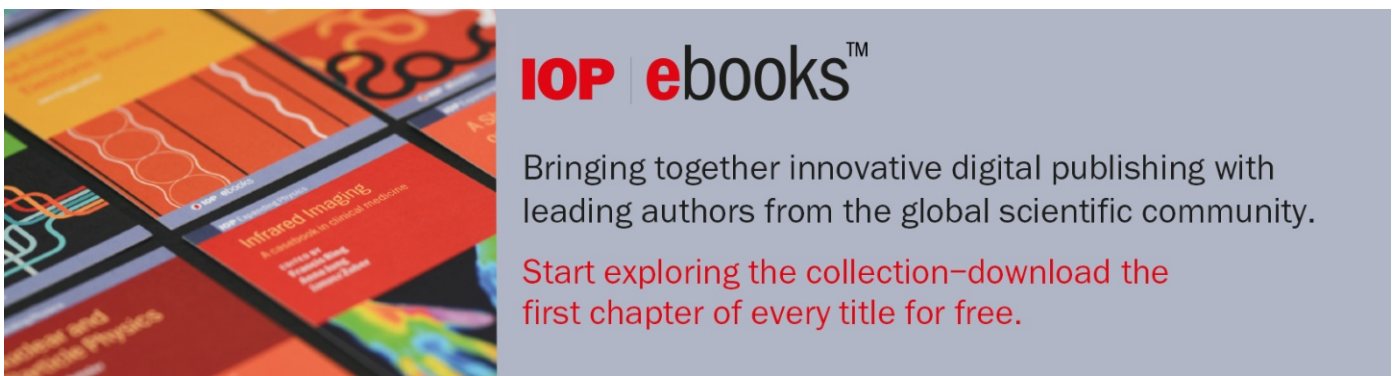


PAPER

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Stability of the inverse source problem for the Helmholtz equation in $\mathbf{R}^3$

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Abstract

We consider the reconstruction of a compactly supported source term in the constant-coefficient Helmholtz equation in $\mathbf{R}^3$, from the measurement of the outgoing solution at a source-enclosing sphere. The measurement is taken at a finite number of frequencies. We explicitly characterize certain finite-dimensional spaces of sources that can be stably reconstructed from such measurements. The characterization involves only the measurement frequencies and the problem geometry parameters. We derive a singular value decomposition of the measurement operator, and prove a lower bound for the spectral bandwidth of this operator. By relating the singular value decomposition and the eigenvalue problem for the Dirichlet–Laplacian on the source support, we devise a fast and stable numerical method for the source reconstruction. We do numerical experiments to validate the stability and efficiency of the numerical method.

Keywords: inverse source problem, Helmholtz equation, singular value decomposition, inverse problems, partial differential equations

(Some figures may appear in colour only in the online journal)

1. Introduction

Assume $R > R_0 > 0$ and let $B_0, B \subset \mathbf{R}^3$ be concentric open balls centered at the origin and of radius R_0 and R , respectively. Let $s \in L^2(B_0)$, and fix a positive frequency (wavenumber) k . Let u_k be the unique outgoing solution of the Helmholtz equation in $\mathbf{R}^3$ with the inhomogeneous term s :

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$$\begin{cases} (\Delta + k^2)u_k &= s \quad \text{in } \mathbf{R}^3, \\ \lim_{|x| \rightarrow \infty} |x|(\partial_{|x|} - ik)u_k &= 0 \quad \text{uniformly for } x/|x| \in S^2. \end{cases} \quad (1)$$

We have

$$u_k(x) = \int_{B_0} G_k(x-y)s(y) dy, \quad x \in \mathbf{R}^3, \quad (2)$$

where G_k is the outgoing fundamental solution of the Helmholtz equation in $\mathbf{R}^3$,

$$G_k(x) = \frac{\exp(ik|x|)}{4\pi|x|}, \quad x \in \mathbf{R}^3 \setminus \{0\}. \quad (3)$$

The forward operator, or measurement operator, F_k associated with the frequency k maps the source to the measurement, $F_k s = u|_{\partial B} = U_k$. The mapping $F_k: L^2(B_0) \rightarrow L^2(\partial B)$ is linear and compact [1]. We call the trace U_k the ‘measurement at frequency k ’. The inverse source problem (ISP) associated with (1) is as follows.

Given the measurements $\{U_{k_j} = u_{k_j}|_{\partial B}\}$ over a set of frequencies $\mathcal{Q} = \{k_j\}$, estimate the source term s in (1).

The multi-frequency ISP arises when the set $\mathcal{Q}$ has more than one element.

The ISP for the Helmholtz equation in $\mathbf{R}^3$ appears, for example, in inverse acoustic scattering and linearized acoustic inverse obstacle and inverse medium problems. Physical applications of this include the design of sound sources with specified radiation patterns, submarine detection by sonar, reconstruction of seismic event history, nondestructive testing of materials, ocean bottom exploration for oil and gas, as well as medical imaging, for example in the form of photo- and thermoacoustic tomography. There is a vast mathematical, physical and engineering literature devoted to the study of the ISP and its applications; see, for instance, [2–8]. Uniqueness, stability and inversion methods for the (multi-frequency) ISP were treated, for example, in [1, 9–20]. The purpose of this work is to investigate the stability of the single-frequency and multi-frequency inverse source problem for the Helmholtz equation in $\mathbf{R}^3$, as well as to present a stable and effective reconstruction method for the multi-frequency problem. The central new technique in our reconstruction method is the explicit and mathematically justified procedure of choosing a sparse set of measurement frequencies that allows accurate solution, given some knowledge of the function space in which the source resides.

In section 2 we derive a singular value system of the forward operator F_k , and in section 3 we use this to give a non-asymptotic spectral characterization of F_k . The main result there is Theorem 2. The analysis in sections 2 and 3 follows our earlier work [14] in $\mathbf{R}^2$ rather closely. Section 4 concerns the multi-frequency ISP, and here we identify certain finite-dimensional spaces of sources that can be stably reconstructed from the boundary measurements. The main result there is Theorem 3, where we give a sufficient criterion for reconstruction of sources belonging to a class of finite-dimensional subspaces of $L^2(B_0)$. In section 5 we conduct numerical experiments to verify our analysis and the reconstruction method, and we finally offer our conclusions and suggestions for further work in section 6.

1.1. Remark

Our analysis is applicable to more general measurement boundaries than the sphere, since in certain cases one may extend a measurement continuously to the sphere. More precisely, let

Γ be the boundary of some C^2 domain Ω that includes the source support B_0 , and take B to be the smallest ball that includes Ω . Let $g = uk|_\Gamma$ be the measurement of the field u_k on Γ . Then g has a unique extension to a measurement U_k on ∂B that depends continuously on g (see Theorem 3.11 in [4]).

2. A singular system of F_k

Our results on the stability of solution of the multi-frequency inverse source problem rely on detailed spectral analysis of the measurement operator F_k . Thus, we first describe a singular value decomposition of F_k in terms of spherical harmonics and spherical Bessel functions. Let $h_m^{(1)}$ be the spherical Hankel function of the first kind and order m , $j_m(z)$ the spherical Bessel function of order m , and $Y_m^n, m \in \mathbb{N}_0, n = -m, \dots, m$, the spherical harmonics. Write $\kappa_0 = kR_0$ and $\kappa = kR$, and let

$$\begin{aligned} a_m(\kappa_0) &= \frac{2\sqrt{k/\pi}}{R_0} \left(\int_0^{R_0} j_m(kr)^2 r^2 dr \right)^{1/2} = \frac{2\sqrt{k/\pi}}{R_0} \left(\int_0^{R_0} r^2 \left(\sqrt{\frac{\pi}{2kr}} J_{m+1/2}(kr) \right)^2 dr \right)^{1/2} \\ &= \frac{\sqrt{2}}{R_0} \left(\int_0^{R_0} r J_{m+1/2}(kr)^2 dr \right)^{1/2} \\ &= \left(J_{m+1/2}(\kappa_0)^2 + J_{m-1/2}(\kappa_0)^2 - \frac{2m+1}{\kappa_0} J_{m+1/2}(\kappa_0) J_{m-1/2}(\kappa_0) \right)^{1/2} \\ &= \left(J_{m+1/2}(\kappa_0)^2 + J_{m+3/2}(\kappa_0)^2 - \frac{2m+1}{\kappa_0} J_{m+1/2}(\kappa_0) J_{m+3/2}(\kappa_0) \right)^{1/2}, \end{aligned} \quad (4)$$

where we use the well-known relation $j_m(z) = \sqrt{\pi/2z} J_{m+1/2}(z)$. The last equality in (4) follows after using the recurrence relation

$$J_{m-1/2}(\kappa_0) = \frac{2(m+1/2)}{\kappa_0} J_{m+1/2}(\kappa_0) - J_{m+3/2}(\kappa_0).$$

For positive arguments, the function j_m is real-valued and has zeros over a subset of the real axis with measure zero. Therefore, we have for any $m \in \mathbb{N}_0$ and any positive κ_0 that

$$a_m(\kappa_0) = 2\sqrt{k/\pi} R_0^{-1} \left(\int_0^{R_0} j_m(kr)^2 r^2 dr \right)^{1/2} > 0.$$

Theorem 1. *The measurement operator F_k admits the singular value decomposition*

$$F_k = \sum_{m \in \mathbb{N}_0} \sum_{n=-m}^m \sigma_m^k(\cdot, u_{m,n}^k)_{L^2(B_0)} v_{m,n}^k \quad (5)$$

with

$$\text{the singular values } \sigma_m^k = \frac{RR_0\sqrt{k\pi}}{2} |h_m^{(1)}(\kappa)| a_m(\kappa_0), \quad (6)$$

$$\text{the right-singular vectors } u_{m,n}^k(x) = \frac{2\sqrt{k/\pi}}{R_0 a_m(\kappa_0)} j_m(k|x|) Y_m^n(x/|x|), \quad x \in B_0, \quad \text{and} \quad (7)$$

$$\text{the left-singular vectors } v_{m,n}^k(x) = (i/R) e^{i \arg h_m^{(1)}(\kappa)} Y_m^n(x/|x|), \quad x \in \partial B. \quad (8)$$

The m th singular value σ_m^k has the multiplicity $2m + 1$.

Proof. The fundamental solution (3) of the Helmholtz equation in $\mathbf{R}^3$ has an expansion in terms of the spherical harmonics [4, Theorem 2.11, p 31]:

$$G_k(x - y) = ik \sum_{m=0}^{\infty} \sum_{n=-m}^m h_m^{(1)}(k|x|) Y_m^n(x/|x|) j_m(k|y|) \overline{Y_m^n(y/|y|)}, \quad |x| > |y|.$$

The L^2 -adjoint $F_k^*: L^2(\partial B) \rightarrow L^2(B_0)$ of F_k is given by

$$F_k^* U(x) = \int_{\partial B} \overline{G_k(x - y)} U(y) dS(y), \quad U \in L^2(\partial B), \quad x \in B_0. \quad (9)$$

Since $\overline{B_0} \subset B$, we have $|z| > |x|$ whenever $z \in \partial B$ and $x \in B_0$. Also, $\{Y_m^n\}$ is an orthonormal basis of $L^2(S^2)$, and $G_k(x - y) = G_k(y - x)$, so for every $s \in L^2(B_0)$ and $x \in B_0$ it holds that

$$\begin{aligned} F_k^* F_k s(x) &= \int_{\partial B} \overline{G_k(z - x)} \int_{B_0} G_k(z - y) s(y) dy dS(z) \\ &= \int_{B_0} s(y) \int_{\partial B} \overline{G_k(z - x)} G_k(z - y) dS(z) dy \\ &= - \int_{B_0} s(y) \int_{\partial B} ik \sum_{m=0}^{\infty} \sum_{n=-m}^m \overline{h_m^{(1)}(k|z|) Y_m^n(z/|z|) j_m(k|x|) Y_m^n(x/|x|)} \\ &\quad \times ik \sum_{\mu=0}^{\infty} \sum_{\nu=-\mu}^{\mu} h_{\mu}^{(1)}(k|z|) Y_{\mu}^{\nu}(z/|z|) j_{\mu}(k|y|) \overline{Y_{\mu}^{\nu}(y/|y|)} dS(z) dy \\ &= \kappa^2 \sum_{m=0}^{\infty} \sum_{n=-m}^m |h_m^{(1)}(\kappa)|^2 j_m(k|x|) Y_m^n(x/|x|) \int_{B_0} s(y) j_m(k|y|) \overline{Y_m^n(y/|y|)} dy. \end{aligned}$$

This is, in fact, an eigendecomposition of the operator $F_k^* F_k$:

$$F_k^* F_k = \sum_{m \in \mathbb{N}_0} \sum_{n=-m}^m (\sigma_{m,n}^k)^2 (\cdot, u_{m,n}^k)_{L^2(B_0)} u_{m,n}^k, \quad (10)$$

with (normalized) eigenvectors

$$u_{m,n}^k(x) = \frac{j_m(k|x|)Y_m^n(x/|x|)}{\|j_m(k|x|)Y_m^n(x/|x|)\|_{L^2(B_0)}}, \quad x \in B_0, m \in \mathbf{N}_0, n = -m, \dots, m,$$

and eigenvalues

$$(\sigma_{m,n}^k)^2 = \kappa^2 |h_m^{(1)}(\kappa)|^2 \|j_m(k|x|)Y_m^n(x/|x|)\|_{L^2(B_0)}^2, \quad m \in \mathbf{N}_0, n = -m, \dots, m.$$

Now each spherical harmonic Y_m^n has unit norm in $L^2(S^2)$, so

$$\begin{aligned} \|j_m(k|x|)Y_m^n(x/|x|)\|_{L^2(B_0)}^2 &= \int_{B_0} |j_m(k|x|)Y_m^n(x/|x|)|^2 dx \\ &= \int_0^{R_0} \int_0^{2\pi} \int_0^\pi j_m(kr)^2 |Y_m^n(\theta, \varphi)|^2 r^2 \sin\theta d\theta d\varphi dr \\ &= \int_0^{R_0} r^2 j_m(kr)^2 dr = \frac{R_0^2 \pi}{4k} a_m(\kappa_0)^2. \end{aligned}$$

Hence

$$u_{m,n}^k(x) = \frac{2\sqrt{k/\pi}}{R_0 a_m(\kappa_0)} j_m(k|x|)Y_m^n(x/|x|), \quad x \in B_0, m \in \mathbf{N}_0, n = -m, \dots, m,$$

and

$$(\sigma_{m,n}^k)^2 = \frac{kR^2 R_0^2 \pi}{4} |h_m^{(1)}(\kappa)|^2 a_m(\kappa_0)^2, \quad m \in \mathbf{N}_0, n = -m, \dots, m.$$

In particular, the eigenvalues $(\sigma_{m,n}^k)^2$ of $F_k^* F_k$ are degenerate with the degree of degeneracy $2m + 1$. In the following, we write $(\sigma_m^k)^2$ for these eigenvalues. By definition, the operator F_k admits the singular values σ_m^k , as in (6). By Theorem 4.7 and its proof in [4, pp 100–101] we have the associated right-singular vectors and left-singular vectors of F_k given respectively by $u_{m,n}^k$, as in (7), and by

$$\begin{aligned} v_{m,n}^k(x) &= \frac{1}{\sigma_{m,n}^k} F_k u_{m,n}^k(x) = \frac{4}{\pi R_0^2 R |h_m^{(1)}(\kappa)| a_m(\kappa_0)^2} \int_{B_0} G_k(x-y) j_m(k|y|) Y_m^n(y/|y|) dy \\ &= \frac{4ik}{\pi R_0^2 R |h_m^{(1)}(\kappa)| a_m(\kappa_0)^2} \sum_{\mu=0}^{\infty} \sum_{\nu=-\mu}^{\mu} h_{\mu}^{(1)}(\kappa) Y_{\mu}^{\nu}(x/|x|) \\ &\quad \times \int_{B_0} j_m(k|y|) j_{\mu}(k|y|) Y_m^n(y/|y|) \overline{Y_{\mu}^{\nu}(y/|y|)} dy \\ &= \frac{4ik \exp(i \arg h_m^{(1)}(\kappa))}{\pi R_0^2 R a_m(\kappa_0)^2} Y_m^n(x/|x|) \int_{r=0}^{R_0} r^2 j_m(kr)^2 dr \\ &= \frac{i \exp(i \arg h_m^{(1)}(\kappa))}{R} Y_m^n(x/|x|), \quad x \in \partial B, \end{aligned}$$

as in (8). □

Remark 1. In [14, Lemma 1, p 4] we followed Bao *et al* [1] and found the following expression for singular values of the measurement operator for the Helmholtz equation in the plane, at frequency k :

$$\sigma_{2D,m}^k = \sqrt{2R\pi R_0} |H_m^{(1)}(\kappa)| A_m(\kappa_0), \quad m \in \mathbf{N}_0.$$

Here $H_m^{(1)} = J_m + iY_m$ is the Hankel function of the first kind and order m , Y_m is the Bessel function of the second kind (also called the Neumann function) of order m , and

$$A_m(\kappa_0) = \left(J_m(\kappa_0)^2 + J_{m+1}(\kappa_0)^2 - \frac{2m}{\kappa_0} J_m(\kappa_0) J_{m+1}(\kappa_0) \right)^{1/2}, \quad m \in \mathbf{N}_0.$$

Since the Bessel functions are defined for all real orders, we may extend the expression for $\sigma_{2D,m}^k$ to $m > 0$ and use the relation $h_m^{(1)}(z) = \sqrt{\pi/2z} H_{m+1/2}^{(1)}(z)$ to express the singular values (6) by

$$\sigma_m^k = \frac{1}{4} \sigma_{2D, m+1/2}^k, \quad m \in \mathbf{N}_0. \quad (11)$$

3. Spectral bandwidth of F_k

As we did in [14], we here define the *bandwidth* $\mathcal{B}$ of the operator $F_k: L^2(B_0) \rightarrow L^2(\partial B)$ to be

$$\mathcal{B} = \arg \min_{m \in \mathbf{N}_0} \{ \sigma_{m+n}^k > \sigma_{m+n+1}^k \quad \text{for all } n \in \mathbf{N}_0 \},$$

that is, the smallest singular value index after which the singular values, ordered as in Theorem 1, are strictly decaying. In [14] we proved and validated numerically an explicit tight lower bound for the bandwidth of F_k in $\mathbf{R}^2$, as well as argued analytically and numerically for a conjectured tight upper bound. In [15] we used the lower bound from [14] to ensure stability of solution of the multi-frequency problem in $\mathbf{R}^2$. We now obtain a lower bound for the bandwidth $\mathcal{B}$ of F_k in $\mathbf{R}^3$ analogous to that in the two-dimensional case. First we need a technical result; let $m \in \mathbf{N}_0$.

Lemma 1. *If $j_\xi(\kappa_0) = 0$ for some $\xi \in [m, m+1]$ then $a_m(\kappa_0) \leq a_{m+1}(\kappa_0)$.*

Proof. Since $j_m(z) = \sqrt{\pi/2z} J_{m+1/2}(z)$, we have from (4) that

$$\begin{aligned} a_m(\kappa_0) &= \sqrt{J_{m+1/2}(\kappa_0)^2 + J_{m-1/2}(\kappa_0)^2 - \frac{2m+1}{\kappa_0} J_{m+1/2}(\kappa_0) J_{m-1/2}(\kappa_0)} \\ &= \sqrt{\frac{2\kappa_0}{\pi}} \sqrt{j_m(\kappa_0)^2 + j_{m-1}(\kappa_0)^2 - \frac{2m+1}{\kappa_0} j_m(\kappa_0) j_{m-1}(\kappa_0)}. \end{aligned}$$

Omitting, for simplicity, the arguments (κ_0) , and using the well-known recurrence relation

$$j_{m-1} + j_{m+1} = \sqrt{\frac{\pi}{2\kappa_0}} (J_{m-1/2} + J_{m+3/2}) = \sqrt{\frac{\pi}{2\kappa_0}} \frac{2(m+1/2)}{\kappa_0} J_{m+1/2} = \frac{2m+1}{\kappa_0} j_m,$$

we find

$$\begin{aligned}
\frac{\pi}{2\kappa_0}(a_m^2 - a_{m+1}^2) &= j_{m-1}^2 - j_{m+1}^2 - \frac{2m+1}{\kappa_0}j_m j_{m-1} + \frac{2m+3}{\kappa_0}j_m j_{m+1} \\
&= (j_{m-1} - j_{m+1})(j_{m-1} + j_{m+1}) - \frac{2m+1}{\kappa_0}j_m(j_{m-1} - j_{m+1}) + \frac{2}{\kappa_0}j_m j_{m+1} \\
&= \frac{2m+1}{\kappa_0}j_m(j_{m-1} - j_{m+1}) - \frac{2m+1}{\kappa_0}j_m(j_{m-1} - j_{m+1}) + \frac{2}{\kappa_0}j_m j_{m+1} \\
&= \frac{2}{\kappa_0}j_m j_{m+1}.
\end{aligned}$$

It was shown in Lemma 2 in [14] that if $\mu_2 > \mu_1 > 0$ and $J_{\mu_1}(\kappa_0) = J_{\mu_2}(\kappa_0) = 0$ then $\mu_2 - \mu_1 > 1$. This result implies that if $\mu_2 > \mu_1 > 0$ and $j_{\mu_1}(\kappa_0) = j_{\mu_2}(\kappa_0) = 0$ then $\mu_2 - \mu_1 > 1$. Thus, if $j_\xi(\kappa_0) = 0$ for some $\xi \in [m, m+1]$ then the function $\mathbf{R} \ni \mu \mapsto j_\mu(\kappa_0)$ equals zero at m , or it equals zero at $m+1$, or it changes sign exactly once in the interval $[m, m+1]$. In either case, we have

$$a_m^2 - a_{m+1}^2 = \frac{4}{\pi}j_m j_{m+1} \leq 0. \quad (12)$$

and it remains to recall that $a_m(\kappa_0) > 0$ for all $m \in \mathbf{N}_0$ and $\kappa_0 > 0$. $\square$

Lemma 2. The function $m \mapsto |h_m^{(1)}(\kappa)|$ is strictly increasing for $m > -1/2$ and $\kappa > 0$.

Proof. The argument is completely analogous to that in remark 1 in [14]. By Nicholson's integral theorem [21, pp 441–44], we have

$$\begin{aligned}
|h_m^{(1)}(\kappa)|^2 &= |j_m(\kappa) + iy_m(\kappa)|^2 = j_m(\kappa)^2 + y_m(\kappa)^2 \\
&= \frac{\pi}{2\kappa} (J_{m+1/2}(\kappa)^2 + Y_{m+1/2}(\kappa)^2) = \frac{\pi}{2\kappa} |H_{m+1/2}^{(1)}(\kappa)|^2 \\
&= \frac{4}{\pi\kappa} \int_0^\infty \cosh\left(2\left(m + \frac{1}{2}\right)t\right) K_0(2\kappa \sinh t) dt,
\end{aligned}$$

where K_0 is the modified Bessel function of the second kind. Differentiating w.r.t. order, we get

$$\partial_m |h_m^{(1)}(\kappa)|^2 = \frac{8}{\pi\kappa} \left(m + \frac{1}{2}\right) \int_0^\infty \sinh\left(2\left(m + \frac{1}{2}\right)t\right) K_0(2\kappa \sinh t) dt,$$

and since K_0 and the hyperbolic sine are positive for positive arguments, it holds that

$$\partial_m |h_m^{(1)}(\kappa)|^2 > 0 \quad \text{for all } m > -1/2.$$

This implies that $|h_m^{(1)}(\kappa)|$ is strictly increasing for $m > -1/2$ and $\kappa > 0$. $\square$

For $m \geq 0$ and $n \in \mathbf{N}$, let $j_{m+1/2,n}$ be the n th positive zero of the Bessel function $J_{m+1/2}$, that is, the n th positive zero of the spherical Bessel function j_m .

Theorem 2. A lower bound $\mathcal{B}_-$ for $\mathcal{B}$ is given by $\mathcal{B}_- = \arg \min_{m \in \mathbf{N}_0} \{j_{m+1/2,1} \geq \kappa_0\}$.

Proof. Let $m \in \mathbf{N}_0$. If $j_{m+1/2,1} < \kappa_0$ then, due to the well-known continuity and strictly increasing behavior of the function $\mu \mapsto j_{\mu,1}$, the value κ_0 is the first positive zero of some

Bessel function $J_{\xi+1/2}$ with $\xi > m$, hence it is the first positive zero of some spherical Bessel function j_ξ with $\xi > m$. There are thus $n \in \mathbf{N}_0$ and $\xi \in [m+n, m+n+1]$ satisfying $j_\xi(\kappa_0) = 0$, and, by Lemma 1, we have $a_{m+n}(\kappa_0) \leq a_{m+n+1}(\kappa_0)$. Also, by Lemma 2, the function $\mathbf{N}_0 \ni \mu \mapsto |h_\mu^{(1)}(\kappa)|$ is strictly increasing. Since the singular value σ_μ from (6) is proportional to $|h_\mu^{(1)}(\kappa)|a_\mu(\kappa_0)$ for $\mu \in \mathbf{N}_0$, we have $\sigma_{m+n} \leq \sigma_{m+n+1}$, hence $m < \mathcal{B}$. In conclusion,

$$\mathcal{B} \geq \arg \max_{m \in \mathbf{N}_0} \{j_{m+1/2, 1} < \kappa_0\} + 1 = \arg \min_{m \in \mathbf{N}_0} \{j_{m+1/2, 1} \geq \kappa_0\}.$$

□

In [14] we found a lower bound $\mathcal{B}_{2D,-}$ for the two-dimensional case to be

$$\mathcal{B}_{2D,-} = \arg \min_{m \in \mathbf{N}_0} \{j_{m,1} \geq \kappa_0\}.$$

In light of the strictly increasing behavior of the function $\mu \mapsto j_{\mu,1}$ for nonnegative orders μ , we interpret the additional ‘+1/2’ in the order of the zero in the three-dimensional case as an indication of greater stability of the inverse source problem in $\mathbf{R}^2$ compared to the three-dimensional case. However, other aspects of the change of setting from $\mathbf{R}^2$ to $\mathbf{R}^3$ may offset this. We finally note that it might be possible to exploit the simple connection (11) between the singular values of the measurement operator in $\mathbf{R}^2$ and in $\mathbf{R}^3$ in an analysis along the lines of [14, pp 9–10] and potentially obtain estimates of tight upper bounds on the bandwidths of both near-field and far-field measurement operators in $\mathbf{R}^3$. We defer this to future work, in particular since it suffices to use the lower bound $\mathcal{B}_-$ in our stability analysis of the multi-frequency inverse source problem below.

With the singular value decomposition and bandwidth results for the measurement operator in place, we begin the analysis of the multi-frequency inverse source problem.

4. Stability of the multi-frequency inverse source problem in $\mathbf{R}^3$

It is well-known [9] that the multi-frequency inverse source problem, as described in our introduction and in [9, problem 3.1], is uniquely solvable only if measurements are available over a set of frequencies with an accumulation point, possibly at infinity. In practical settings, it is impossible to perform infinitely many measurements, so we turn to reconstructing sources in finite-dimensional subspaces of $L^2(B_0)$. The analysis in this section is a natural extension of the methods in [15] from $\mathbf{R}^2$ to $\mathbf{R}^3$, formulated in a simpler and more flexible way.

4.1. Finite-dimensional source spaces

Consider first the eigenvalue problem for the Dirichlet–Laplacian in B_0 ,

$$\begin{aligned} -\Delta \varphi &= \lambda \varphi & \text{in } B_0, \\ \varphi &= 0 & \text{on } \partial B_0. \end{aligned} \tag{13}$$

One can check [22] that the eigenfunctions and eigenvalues are given uniquely up to a scaling by

$$\varphi_{\alpha,\beta,\gamma} = \sqrt{\frac{2}{R_0^3}} \frac{j_\alpha(\mu_{\alpha,\beta}|x|/R_0)}{|j_{\alpha+1}(\mu_{\alpha,\beta})|} Y_\alpha^\gamma(x/|x|), \quad \alpha \in \mathbf{N}_0, \beta \in \mathbf{N}, \gamma \in \{-\alpha, \dots, \alpha\}, \tag{14}$$

$$\lambda_{\alpha,\beta} = \left(\frac{\mu_{\alpha,\beta}}{R_0} \right)^2, \quad \alpha \in \mathbf{N}_0, \beta \in \mathbf{N}, \tag{15}$$

where $\mu_{\alpha,\beta}$ is the β th positive zero of the spherical Bessel function of order α , i.e. $j_\alpha(\mu_{\alpha,\beta}) = 0$ for $\beta \in \mathbb{N}^4$. Each eigenvalue $\lambda_{\alpha,\beta}$ has multiplicity $2\alpha + 1$.

It is well-known that the set $\{\varphi_{\alpha,\beta,\gamma}\}$ of normalized eigenfunctions is an orthonormal basis of $L^2(B_0)$. Any source $s \in L^2(B_0)$ thus has an eigenfunction expansion

$$s(x) = \sum_{\alpha,\beta,\gamma} \hat{s}_{\alpha,\beta,\gamma} \varphi_{\alpha,\beta,\gamma}(x), \quad \hat{s}_{\alpha,\beta,\gamma} = \int_{B_0} s \overline{\varphi}_{\alpha,\beta,\gamma} dx. \quad (16)$$

For $M \in \mathbb{N}$, let $\Lambda_M = \{\lambda_{\alpha,\beta}\}$ be a set of any M eigenvalues in (15). We associate with Λ_M the finite-dimensional source space

$$S_{\Lambda_M}(B_0) = \text{span}\{\varphi_{\alpha,\beta,\gamma} : \lambda_{\alpha,\beta} \in \Lambda_M, \quad -\alpha \leq \gamma \leq \alpha\} \subset L^2(B_0). \quad (17)$$

Due to the multiplicity of the eigenvalues, it follows that $M \leq \dim S_{\Lambda_M}(B_0) \leq M(1 + 2A)$, with $A = \max\{\alpha, \lambda_{\alpha,\beta} \in \Lambda_M\}$. A natural example of such finite-dimensional source space is given by the sources of the form

$$s = \sum_{\alpha=0}^{N_1} \sum_{\beta=1}^{N_2} \sum_{\gamma=-\alpha}^{\alpha} \hat{s}_{\alpha,\beta,\gamma} \varphi_{\alpha,\beta,\gamma}. \quad (18)$$

Then $\Lambda_M = \{\lambda_{\alpha,\beta}\}_{\alpha=0,\dots,N_1, \beta=1,\dots,N_2}$, written $\{\lambda_{\alpha,\beta}\}_{\alpha=0, \beta=1}^{N_1, N_2}$ in the following, and hence $M = N_2(N_1 + 1)$ and $\dim S_{\Lambda_M}(B_0) = N_2(N_1 + 1)^2$.

As we will see, the spaces $S_{\Lambda_M}(B_0)$ allow us to relate in a constructive manner the source information contained in the measurements at different frequencies to a common basis. We now address the following questions.

- Given a finite-dimensional source space $S_{\Lambda_M}(B_0)$, can we identify a set of measurement frequencies $Q = \{k_j\}$ such that $|Q| \ll \dim S_{\Lambda_M}(B_0)$ and such that any $s \in S_{\Lambda_M}(B_0)$ can be reconstructed exactly from measurements at these frequencies?
- Vice versa: given a finite set of measurement frequencies Q , can we identify the finite-dimensional source spaces $S_{\Lambda_{M_j}}(B_0)$ in which the sources can be reconstructed exactly from measurements at these frequencies?
- Will such reconstructions be stable with respect to imperfect or noisy measurements? By ‘stable’ we mean in general that the noise in the measurements is not unacceptably amplified in the reconstructed source. More precisely, for a fixed frequency k , let $U_k^\varepsilon = U_k + \varepsilon$ be a noisy measurement, where $\varepsilon \in L^2(\partial B)$ represents an additive noise term. In our reconstruction method, we will need to compute certain functionals of the measurements of the form

$$\hat{U} = \frac{(U_k^\varepsilon, v_j)_{L^2(\partial B)}}{\sigma_j^k} = \frac{(U_k, v_j)_{L^2(\partial B)}}{\sigma_j^k} + \frac{(\varepsilon, v_j)_{L^2(\partial B)}}{\sigma_j^k}.$$

The noise component in this expression will be amplified by the factor $1/\sigma_j^k$, which is unbounded as $j \rightarrow \infty$. Our reconstruction method is stable because we control the maximum value of $1/\sigma_j^k$ used in the reconstruction to avoid unacceptable amplification of such additive noise.

⁴For clarity we here use the notation $\mu_{\alpha,\beta} = j_{\alpha+1/2,\beta}$.

The main results, and answers to the above questions, are summarized as follows.

Theorem 3. *Let $s \in S_{\Lambda_M}(B_0)$, for some finite set Λ_M of eigenvalues. There is a $\Delta k \in (0, 1/R_0]$ such that, if for each $\lambda_{\alpha,\beta} \in \Lambda_M$ there is a measurement frequency $k_j \in Q$ satisfying $|k_j - \lambda_{\alpha,\beta}^{1/2}| < \Delta k$, then the source s can be stably reconstructed from the set of measurements $\{U_{k_j}\}_{k_j \in Q}$.*

Corollary 1. *Let Q be a finite set of measurement frequencies. There is a $\Delta k \in (0, 1/R_0]$ such that for any set of eigenvalues Λ_{M_j} satisfying $\min_{k_j \in Q} |\lambda_{\alpha,\beta} - k_j| < \Delta k$ for all $\lambda_{\alpha,\beta} \in \Lambda_{M_j}$, any source $s \in S_{\Lambda_{M_j}}(B_0)$ can be stably reconstructed from the measurements $\{U_{k_j}\}_{k_j \in Q}$.*

4.1.1. Remarks.

- Theorem 3 says that if we have measurements at a set of frequencies Q that is close enough to the set Λ_M of eigenvalues, then any source $s \in S_{\Lambda_M}(B_0)$ can be stably reconstructed from these measurements. Conversely, given a set Q of measurement frequencies, we can find sets of eigenvalues satisfying the hypothesis in Theorem 3. These sets then give rise to finite-dimensional source spaces, the elements of which are stably reconstructible. We will later also show that, due to the distribution of the eigenvalues of the Dirichlet Laplacian on B_0 , a single frequency k can be close to multiple eigenvalues. Consequently, sources from high-dimensional $S_{\Lambda_M}(B_0)$ are reconstructible even from quite few measurement frequencies.
- If $s \notin S_{\Lambda_M}(B_0)$, we reconstruct an approximation $\tilde{s} = P_{S_{\Lambda_M}(B_0)}s$, the projection of s onto $S_{\Lambda_M}(B_0)$. The quality of the approximation depends on $\|\tilde{s} - s\|_{L^2(B_0)}$, and hence on $S_{\Lambda_M}(B_0)$.
- Given $S_{\Lambda_M}(B_0)$, we can extract a (not necessarily unique) smallest set of measurement frequencies Q_m sufficient for any $s \in S_{\Lambda_M}(B_0)$ to be stably reconstructed from measurements $\{U_{k_j}\}_{k_j \in Q_m}$. However, our analysis does not provide a necessary criterion for such a set Q_m , and hence it might be possible to reconstruct sources from even smaller frequency sets.

The above results are achieved by utilizing the special relationship between the singular vectors of the measurement operator F_k and the eigenfunctions of the Dirichlet–Laplacian, summarized in the following lemma.

Lemma 3. *For any positive measurement frequency k we have*

$$(\varphi_{\alpha,\beta,\gamma}, u_{m,n}^k)_{L^2(B_0)} = \delta_{\alpha,m} \delta_{\gamma,n} \operatorname{sign}(j_{m+1}(\mu_{m,\beta})) \frac{\sqrt{8k/\pi R_0} j_m(\kappa_0)}{a_m(\kappa_0)} \frac{\lambda_{m,\beta}^{1/2}}{\lambda_{m,\beta} - k^2}, \quad (19)$$

where $\delta_{m,n}$ is the Kronecker delta. In addition, if $k = \lambda_{m,i}^{1/2}$ for some m, i , then the right-singular vectors of the measurement operator F_k are the eigenvectors of the Dirichlet Laplacian: $u_{m,n}^k = \varphi_{m,i,n}$, for each $n \in [-m, \dots, m]$.

Proof. From the orthonormality of the spherical harmonics, we immediately have that

$$(\varphi_{\alpha,\beta,\gamma}, u_{m,n}^k)_{L^2(B_0)} = \delta_{\alpha,m} \delta_{\gamma,n} \frac{\sqrt{8k/\pi R_0^5}}{|j_{\alpha+1}(\mu_{\alpha,\beta})| a_m(\kappa_0)} \int_0^{R_0} j_\alpha(\mu_{\alpha,\beta} r/R_0) j_m(kr) r^2 dr.$$

Setting $\alpha = m$, evaluating the above integral and using the recurrence relations for Bessel functions, as well as that $\lambda_{\alpha,\beta} = (\mu_{\alpha,\beta}/R_0)^2$, we get

$$\int_0^{R_0} j_m(\mu_{m,\beta}/R_0 r) j_m(kr) r^2 dr = \frac{R_0^2 \lambda_{m,\beta}^{1/2}}{\lambda_{m,\beta} - k^2} j_m(kR_0) j_{m+1}(\mu_{m,\beta}).$$

This shows the first statement. For the second part we note that, when $z = \mu_{m,i}$, we have

$$a_m(z) = \sqrt{\frac{2z}{\pi}} \left(j_m^2(z) + j_{m+1}^2(z) - \frac{2m+1}{z} j_m(z) j_{m+1}(z) \right)^{1/2} = \sqrt{\frac{2z}{\pi}} |j_{m+1}(z)|.$$

Recall that $\lambda_{m,i}^{1/2} = \mu_{m,i}/R_0$ and $\kappa_0 = kR_0$. Taking $k = \lambda_{m,i}^{1/2}$, we have that

$$\begin{aligned} u_{m,n}^k(x) &= \frac{2\sqrt{k/\pi}}{R_0 a_m(\kappa_0)} j_m(k|x|) Y_m^n(x/|x|) \\ &= \frac{\sqrt{2}}{R_0^{3/2} |j_{m+1}(\mu_{m,i})|} j_m(k|x|) Y_m^n(x/|x|) \\ &= \varphi_{m,i,n}(x), \quad m \in \mathbf{N}_0, n \in \{-m, \dots, m\}. \end{aligned}$$

□

To prove Theorem 3 we also need the following result.

Lemma 4. For any numbers $k_1, \dots, k_N$ satisfying $0 < k_1 < \dots < k_N$ there exists a largest $\Delta k > 0$ such that for any $1 \leq i \leq N$ and any real x , $|k_i - x| < \Delta k$ implies

$$\left| \frac{k_i}{k_i^2 - x^2} \right| > \sum_{j=1, j \neq i}^N \left| \frac{k_j}{k_j^2 - x^2} \right|.$$

Proof. For any index i , assume $x \in [k_i - \Delta k, k_i + \Delta k] \subset (k_{i-1}, k_{i+1})$, with the convention that $k_0 = 0$ and $k_{N+1} > k_N$. The function

$$S(x) = \sum_{j=1, j \neq i}^N \left| \frac{k_j}{k_j^2 - x^2} \right|, \quad x \in [k_i - \Delta k, k_i + \Delta k],$$

is convex and has a maximum M at one of the endpoints $k_i \pm \Delta k$. Next, the function

$$D(x) = \left| \frac{k_i}{k_i^2 - x^2} \right|, \quad x \in [k_i - \Delta k, k_i + \Delta k],$$

is unbounded at k_i , and it is even. We can therefore find a largest Δk such that for all $x \in (k_i - \Delta k, k_i + \Delta k)$, we have $D(x) > M \geq S(x)$. □

We now construct the mapping between the multi-frequency measurements and the finite-dimensional source space. Let Λ_M be a set of M eigenvalues and Q a finite set of measurement frequencies, with $|Q| \leq M$. For the sake of bookkeeping, we introduce some new notation.

- (a) The auxiliary frequency set Q_M : for each $\lambda_{\alpha,\beta} \in \Lambda_M$, take the corresponding $k_{\alpha,\beta} \in Q_M$ to be $k_{\alpha,\beta} = \arg \min_{k_j \in Q} |\lambda_{\alpha,\beta}^{1/2} - k_j|$, i.e. the measurement frequency closest to $\lambda_{\alpha,\beta}^{1/2}$. Then Q_M is a set of M not necessarily distinct numbers from Q .

- (b) We label a measurement at frequency $k_{\alpha,\beta} \in Q_M \subset Q$ as $U_{\alpha,\beta}$. That is, if $k_{\alpha,\beta} = k_j \in Q$, then $U_{\alpha,\beta} = U_{k_j}$.
- (c) Let I_{Λ_M} be the increasingly ordered set of indices α such that $\lambda_{\alpha,\beta} \in \Lambda_M$. For each $\alpha \in I_{\Lambda_M}$, we define the set $I_\alpha = \{\beta' \in \mathbf{N}: \lambda_{\alpha,\beta'} \in \Lambda_M\}$, also increasingly ordered.
If for example $\Lambda_4 = \{\lambda_{1,2}, \lambda_{3,1}, \lambda_{3,3}, \lambda_{7,3}\}$, we have that $I_{\Lambda_4} = \{1, 3, 7\}$, and that $I_1 = \{2\}$, $I_3 = \{1, 3\}$ and $I_7 = \{3\}$.
- (d) We associate to each measurement $U_{\alpha,\beta}$ a set of coefficients:

$$\hat{u}_{\alpha,\beta,\gamma} = \frac{(U_{\alpha,\beta}, v_{\alpha,\gamma}^{k_{\alpha,\beta}})_{L^2(\partial B)}}{\sigma_{\alpha}^{k_{\alpha,\beta}}}, \quad k_{\alpha,\beta} \in Q_M, \gamma \in \{-\alpha, \dots, \alpha\}. \quad (20)$$

Here $v_{\alpha,\beta}^{k_{\alpha,\beta}}$ and $\sigma_{\alpha}^{k_{\alpha,\beta}}$ are the singular vectors and singular values of $F_{k_{\alpha,\beta}}$, respectively. We now construct a family of matrix operators mapping the eigenfunction expansion coefficients $\{\hat{s}_{\alpha,\beta,\gamma}\}$ of a source $s \in S_{\Lambda_M}(B_0)$ to the measurement coefficients $\{\hat{u}_{\alpha,\beta,\gamma}\}$. Expressing $s \in S_{\Lambda_M}(B_0)$ in the above notation we have

$$s = \sum_{\alpha \in I_{\Lambda_M}} \sum_{\beta \in I_\alpha} \sum_{\gamma = -\alpha}^{\alpha} \hat{s}_{\alpha,\beta,\gamma} \varphi_{\alpha,\beta,\gamma}.$$

Fixing $m \in I_{\Lambda_M}$ and letting $U_{m,n}$ be the measurement associated with the source s yields

$$\begin{aligned} \hat{u}_{m,n,i} &= \frac{(U_{m,n}, v_{m,i}^{k_{m,n}})_{L^2(\partial B)}}{\sigma_m^{k_{m,n}}} \\ &= (s, u_{m,i}^{k_{m,n}})_{L^2(B_0)} \\ &= \sum_{\beta \in I_m} \hat{s}_{m,\beta,i} (\varphi_{m,\beta,i}, u_{m,i}^{k_{m,n}})_{L^2(B_0)}, \quad n \in I_m, i \in \{-m, \dots, m\}. \end{aligned} \quad (21)$$

This is a linear system relating the source coefficients with the measurement coefficients. Setting $N = |I_m|$, we define

$$\bar{U}_m^i = (\hat{u}_{m,j,i})_{j \in I_m} \in \mathbf{C}^N, \quad (22)$$

$$\bar{S}_m^i = (\hat{s}_{m,j,i})_{j \in I_m} \in \mathbf{C}^N, \quad (23)$$

$$K_m = \left((\varphi_{m,j,i}, u_{m,i}^{k_{m,n}})_{L^2(B_0)} \right)_{n,j \in I_m} \in \mathbf{C}^{N \times N}. \quad (24)$$

Notice that the matrix K_m does not depend on the index i . Equation (21) now takes the form

$$\bar{U}_m^i = K_m \bar{S}_m^i, \quad i \in \{-m, \dots, m\}. \quad (25)$$

To reconstruct all the coefficients of $s \in S_{\Lambda_M}(B_0)$, we thus have to solve the system of equations

$$\bar{U}_m^i = K_m \bar{S}_m^i, \quad \forall m \in I_{\Lambda_M}, i \in \{-m, \dots, m\}, \quad (26)$$

and so we have to show that the matrices K_m are invertible.

To this end, we show that there exists a $\Delta k > 0$ such that if for each $\lambda_{\alpha,\beta} \in \Lambda_M$ there is a corresponding measurement frequency $k_{\alpha,\beta} \in Q_M \subset Q$ close to $\lambda_{\alpha,\beta}$ in the sense that $|\lambda_{\alpha,\beta}^{1/2} - k_{\alpha,\beta}| < \Delta k$, then each matrix K_m is invertible. In addition, we show that if $\Delta k \leq 1/R_0$, then the inversion is stable with respect to noisy measurements.

Invertibility of the matrices K_m is shown by proving that Δk can be chosen such that each matrix K_m is strictly diagonally dominant. Recall that an $M \times M$ matrix $A = (a_{i,j})$ is strictly diagonally dominant if $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$ for each $1 \leq i \leq M$, and that such matrices are invertible [23].

Proof of Theorem 3 (Reconstruction). From equation (24) we get that for K_m , $m \in I_{\Lambda_M}$ to be strictly diagonally dominant, the inequality

$$|(\varphi_{m,n,i}, u_{m,i}^{k_{m,n}})_{L^2(B_0)}| > \sum_{\beta \in I_m, \beta \neq n} |(\varphi_{m,\beta,i}, u_{m,\beta}^{k_{m,n}})_{L^2(B_0)}| \quad (27)$$

has to hold for each row. Recall (21) and that the index $n \in I_m$ is row-dependent. With the help of Lemma 3 we write (27) as

$$\left| \frac{\sqrt{8k/\pi R_0} j_m(\kappa_0)}{a_m(\kappa_0)} \frac{\lambda_{m,n}^{1/2}}{\lambda_{m,n} - k_{m,n}^2} \right| > \sum_{\beta \in I_m, \beta \neq n} \left| \frac{\sqrt{8k/\pi R_0} j_m(\kappa_0)}{a_m(\kappa_0)} \frac{\lambda_{m,\beta}^{1/2}}{\lambda_{m,\beta} - k_{m,n}^2} \right|. \quad (28)$$

The factor $\sqrt{8k/\pi R_0} j_m(\kappa_0)/a_m(\kappa_0)$ is constant in each row and can be neglected. Hence the simplification

$$\left| \frac{\lambda_{m,n}^{1/2}}{\lambda_{m,n} - k_{m,n}^2} \right| > \sum_{\beta \in I_m, \beta \neq n} \left| \frac{\lambda_{m,\beta}^{1/2}}{\lambda_{m,\beta} - k_{m,n}^2} \right|. \quad (29)$$

For fixed m and increasing $\beta \in I_m$, the eigenvalues $\{\lambda_{m,\beta}\}_{\beta \in I_m}$ form a strictly increasing sequence. Assuming that $k_{m,n} \in (\lambda_{m,n-1}^{1/2}, \lambda_{m,n+1}^{1/2})$, Lemma 4 shows that there exists a largest $\Delta k_{m,n} > 0$ such that if $|\lambda_{m,n}^{1/2} - k_{m,n}| < \Delta k_{m,n}$ then the inequality (29) holds. We take Δk_m to be $\Delta k_m = \min_{n \in I_m} \Delta k_{m,n}$, i.e. the smallest of the $\Delta k_{m,n}$'s such that (29) holds for each row in K_m . Next, we set $\Delta k = \min_{m \in I_{\Lambda_M}} \Delta k_m$. Assume now that for each $\lambda_{m,n} \in \Lambda_M$ there is a measurement frequency $k_j \in \mathcal{Q}$ such that $|\lambda_{m,n}^{1/2} - k_j| < \Delta k$. Then, for $k_{m,n} = k_j \in \mathcal{Q}_M$, we immediately have that

$$|\lambda_{m,n}^{1/2} - k_{m,n}| < \Delta k \leq \Delta k_{m,n},$$

and hence the inequality (29) holds for each row in each K_m . Hence each K_m is strictly diagonally dominant and invertible. $\square$

4.1.2. Remark. The above proof finds a single Δk that serves as a ‘global’ constraint on how close it suffices for the measurement frequencies to be to the eigenvalues in Λ_M . This is done to make the statement of Theorem 3 simpler. In a more refined analysis, one could aim to find a Λ_M -dependent Δk . Moreover, the above is merely an existence proof for Δk . Clearly, we want Δk as large as possible (but, as we will see, smaller than or equal to $1/R_0$ for stability reasons). This can of course be checked numerically, by computing the rank or condition number of the matrices K_m as a function of Δk , and doing this reveals that one can choose $\Delta k = 1/R_0$. However, we have not yet been able to obtain a proof of this.

We now analyze the stability of the above inversion procedure with respect to noise. We will see that under the simple constraint that $\Delta k \leq 1/R_0$, where R_0 is the radius of the source domain, we can guarantee stable reconstructions in the sense that noise is not amplified in the calculation of the measurement coefficients $\hat{u}_{m,n,i}$.

For $\varepsilon \in L^2(\partial B)$, we consider an additive noise model where the noisy measurements are of the form $U_{m,n}^\varepsilon = U_{m,n} + \varepsilon$. We assume the noise ε to be bounded in the sense that $\|\varepsilon\|_{L^2(\partial B)} \leq \delta \|U_{m,n,i}\|_{L^2(\partial B)}$ for some noise level $0 < \delta < 1$. Recalling equation (20) for the measurement coefficients,

$$\hat{u}_{m,n,i} = \frac{(U_{m,n}^\varepsilon, v_{m,i}^{k_{m,n}})_{L^2(\partial B)}}{\sigma_m^{k_{m,n}}} = \frac{(U_{m,n}, v_{m,i}^{k_{m,n}})_{L^2(\partial B)}}{\sigma_m^{k_{m,n}}} + \frac{(\varepsilon, v_{m,i}^{k_{m,n}})_{L^2(\partial B)}}{\sigma_m^{k_{m,n}}}, \quad (30)$$

it is evident that the noise will be amplified if $\sigma_m^{k_{m,n}} \ll 1$. Hence we want to avoid the coefficients associated with small singular values, i.e. coefficients $\hat{u}_{m,n,i}$ where $m > \mathcal{B}$, where $\mathcal{B}$ is the bandwidth of the operator $F_{k_{m,n}}$. We now show that if $\Delta k \leq 1/R_0$, then $m \leq \mathcal{B}_-$.

Proof of Theorem 3 (Stability). The result follows from Theorem 2 in [15]. We adapt the proof to the current setting, including all the relevant details for completeness.

To each measurement frequency $k_j \in Q$ we associate the index set

$$I_{k_j} = \{\alpha: \lambda_{\alpha,\beta} \in \Lambda_M \quad \text{and} \quad |\lambda_{\alpha,\beta}^{1/2} - k_j| < \Delta k\}.$$

For each measurement U_{k_j} , we defined I_{k_j} to be the set of indices m of the measurement coefficients that are computed from U_{k_j} . Consequently, by (30), the elements of I_{k_j} are also the indices of the singular values involved in the computation. We show that if $\Delta k \leq 1/R_0$, we have $\max_m I_{k_j} \leq \mathcal{B}_-$. Recalling Theorem 2, we have

$$\mathcal{B}_- = \arg \min_{m \in \mathbb{N}_0} \{j_{m+1/2,1} = \mu_{m,1} \geq \kappa_0 = k_j R_0\}.$$

As shown in [15], for $m > -1$ and $n \in \mathbb{N}$ it holds that $\mu_{m+1,n} - \mu_{m,n} > 1$. Recalling that $\lambda_{m,n} = (\mu_{m,n}/R_0)^2$, we have that

$$|\lambda_{m,n}^{1/2} - k_j| < \Delta k \iff \mu_{m,n} \in (k_j R_0 - 1, k_j R_0 + 1).$$

Assume first that the largest index $\tilde{m} \in I_{k_j}$ is associated with some $\mu_{\tilde{m},1}$ satisfying

$$\mu_{\tilde{m},1} \in (k_j R_0 - 1, k_j R_0 + 1).$$

Then either $\mu_{\tilde{m},1} < k_j R_0$ and so $\tilde{m} < \mathcal{B}_-$, or $\mu_{\tilde{m},1} \in [k_j R_0, k_j R_0 + 1)$ and so $\tilde{m} = \mathcal{B}_-$, since no two $\mu_{m,1}$ can occur within an interval of length 1, and $\mu_{\mathcal{B}_-,1}$ is the smallest $\mu_{m,1} \geq k_j R_0$. Last, assume the largest index $\tilde{m}$ comes from $\mu_{\tilde{m},n}$, $n > 1$, and that $\tilde{m} > \mathcal{B}_-$. Then $\mu_{\tilde{m},n} > \mu_{\tilde{m},1} > \mu_{\mathcal{B}_-,1}$, which is a contradiction since then $\mu_{\tilde{m},n} \notin (k_j R_0 - 1, k_j R_0 + 1)$. $\square$

4.2. Density of eigenvalues and subspace dimension

A key observation in [15] was that the eigenvalues become more densely spaced as one moves up the real line. An estimate for the expected number $N_\lambda(k, \Delta k)$ of $\lambda_{\alpha,\beta}^{1/2}$ that lay in an interval of length $2\Delta k$ centered at k was in [15] given by the formula

$$N_\lambda(k, \Delta k) \approx \left\lfloor \left(\frac{1}{2}(3k^2 + 16)^{1/2} - 3 \right) \right\rfloor \frac{2\Delta k}{R_0\pi}. \quad (31)$$

Roughly, this implies that for a set of N linearly spaced frequencies $Q = \{2\Delta k \cdot n\}_{n=1}^N$, we have

$$|\Lambda| \approx \frac{(\Delta k)^2}{R_0\pi} 3N(N+1),$$

and hence that the cardinality of Λ_M increases approximately quadratically with N . Equation (31) also has the interpretation that high-frequency measurements carry more information in terms of the dimension of $S_{\Lambda_M}(B_0)$.

5. Numerical verification of the multi-frequency inversion method

The proof of Theorem 3 immediately suggests a reconstruction method:

- For a given finite-dimensional source space S_Λ , find a set of measurement frequencies $Q = \{k_j\}$ satisfying $|k_j - \lambda_{\alpha,\beta}^{1/2}| < \Delta k$, as in Theorem 3.
- Compute the measurement coefficients from the measurements $\{U_{k_j}\}_{k_j \in Q}$ as in equation (21).
- Solve the linear systems in (26) to obtain the source coefficients and a reconstruction of the source.

Step (a) above can also be done in the reversed direction, i.e. given a set of measurement frequencies Q , find a space S_{Λ_M} satisfying corollary 1 and in which a stable reconstruction can be achieved.

For numerical experiments and validation of the above reconstruction method, we introduce two finite-dimensional source spaces. Let $\Lambda_L = \{\lambda_{\alpha,\beta}\}_{\alpha=0,\beta=1}^{5,5}$ and $\Lambda_H = \{\lambda_{\alpha,\beta}\}_{\alpha=0,\beta=1}^{15,15}$, and let S_{Λ_L} and S_{Λ_H} be the corresponding source spaces. Further, let the source support B_0 have radius 1, and assume measurements are taken on ∂B , where B has radius 2. By numerical testing, we find that we can choose $\Delta k = 0.7$ in Theorem 3 and we find the smallest (non-unique) sets of measurement frequencies Q_L and Q_H sufficient for stable reconstruction of sources in S_{Λ_L} and S_{Λ_H} , respectively. We get

$$|Q_L| = 12, \quad |\Lambda_L| = 30, \quad \dim S_{\Lambda_L} = 180, \quad (32)$$

$$|Q_H| = 42, \quad |\Lambda_H| = 240, \quad \dim S_{\Lambda_H} = 3840. \quad (33)$$

Due to the density of the eigenvalues, we see that we can reconstruct 3840 source coefficients from measurements at only 42 different frequencies.

Let $\chi_{B_{r,x_0}}(x)$ be the characteristic function for a ball of radius r centered at x_0 in $\mathbf{R}^3$. We consider the two sources

$$s_E(x) = 1\varphi_{2,1,1} + 2\varphi_{3,3,3} + 4\varphi_{4,3,2} + 0.5\varphi_{12,6,7}, \quad (34)$$

$$s_B(x) = 1\chi_{B_{r_1,x_1}}(x) + 2\chi_{B_{r_2,x_2}}(x) + 3/2\chi_{B_{r_3,x_3}}(x), \quad (35)$$

where the radii and centers of the characteristic functions are

$$\begin{aligned} r_1 &= 0.4, & x_1 &= (0, 0.1, 0.3), & r_2 &= 0.1, & x_2 &= (0, -0.4, -0.3), \\ r_3 &= 0.3, & x_3 &= (0.2, 0.5, -0.4). \end{aligned}$$

Clearly, $s_E \in S_{\Lambda_H} \setminus S_{\Lambda_L}$. On the other hand, s_B belongs to neither S_{Λ_L} nor S_{Λ_H} , and we can only reconstruct the projection of the source onto the source spaces.

To compute the measurements $\{U_{k_j}\}_{k_j \in Q}$, we do as follows:

For a source $\chi_{B_{r,x_0}}(x)$ supported in B_0 we have

$$\begin{aligned} F_k \chi_{B_{r,x_0}}(x) &= \int_{B_0} G_k(x, y) \chi_{B_{r,x_0}}(y) dy \\ &= \int_{B_{r,x_0}} \frac{e^{ik|x-y|}}{4\pi|x-y|} dy = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \int_{B_{r,x_0}} |x-y|^{n-1} dy. \end{aligned} \quad (36)$$

The integrals in the sum can be evaluated by noting that the function $|x-y|^{n-1}$ is constant for $y \in \partial B_{z,x}$. Let $d = |x-x_0|$. For $d-r \leq z \leq d+r$ the sphere $\partial B_{z,x}$ intersects B_{r,x_0} , and the part of $\partial B_{z,x}$ enclosed in B_{r,x_0} is a spherical cap with area $A(z) = 2\pi z (z - (d^2 - r^2 + z^2)/2d)$, see [24]. We thus have that

$$\int_{B_{r,x_0}} |x-y|^{n-1} dy = \int_{d-r}^{d+r} z^{n-1} 2\pi z \left(z - \frac{d^2 - r^2 + z^2}{2d} \right) dz. \quad (37)$$

This integral can be evaluated by standard methods and we get

$$\int_{B_{r,x_0}} |x-y|^{n-1} dy = \frac{2\pi ((d+r(2+n))(d-r)^{n+2} + ((n+2)r-d)(d+r)^{n+2})}{d(n+1)(n+2)(n+3)}. \quad (38)$$

A lengthy calculation shows that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(ik)^n}{4\pi} \frac{2\pi ((d+r(2+n))(d-r)^{n+2} + ((n+2)r-d)(d+r)^{n+2})}{dn!(n+1)(n+2)(n+3)} \\ &= \frac{(i-kr)e^{ik(d-r)} - (i+kr)e^{ik(d+r)}}{2dk^3}, \end{aligned}$$

and hence the measurement is given by

$$F_k \chi_{B_{r,x_0}}(x) = \frac{(i-kr)e^{ik(|x-x_0|-r)} - (i+kr)e^{ik(|x-x_0|+r)}}{2|x-x_0|k^3}, \quad x_0 \in B_0, x \in \partial B. \quad (39)$$

The measurement corresponding to an eigenvector source $\varphi_{\alpha,\beta,\gamma}$ is given by (19) in Lemma 3, and we have that

$$F_k \varphi_{\alpha,\beta,\gamma}(x) = \sigma_{\alpha}^k v_{\alpha,\gamma}^k(x) \operatorname{sign}(j_{\alpha+1}(\mu_{\alpha,\beta})) \frac{\sqrt{8k/\pi R_0} j_{\alpha}(\kappa_0)}{a_{\alpha}(\kappa_0)} \frac{\lambda_{\alpha,\beta}^{1/2}}{\lambda_{\alpha,\beta} - k^2}.$$

We avoid the high computational cost of simulating high-frequency solutions to the Helmholtz equation by generating the measurements of the sources using the above closed-form solutions.

We sample each of the measurements $\{U_{k_j}\}_{k_j \in Q}$ at equidistant points $\{x_i\}_{i=1}^N$ on ∂B . For a noise level δ , we add a noise vector $\varepsilon \in \mathbb{C}^N$ of normally distributed random numbers⁵ to the measurement vector $(U_{k_j}(x_i)) \in \mathbb{C}^N$, with the convention that $|\varepsilon| \leq \delta |U_{k_j}|$. We denote the noisy measurement by $U_{k_j}^{\varepsilon} = U_{k_j} + \varepsilon$.

⁵For each component ε_i the real and imaginary part is generated separately by the Matlab-function `randn()`.

To compute the measurement coefficients, we approximate the inner products appearing in (21) by a simple quadrature method

$$\left(U_{k_j}^\varepsilon, v_{\alpha,\gamma}^{k_j} \right)_{L^2(\partial B)} \approx 4\pi R^2 \frac{1}{N} \sum_{i=1}^N U_{k_j}^\varepsilon(x_i) \overline{v_{\alpha,\gamma}^{k_j}(x_i)}. \quad (40)$$

All entries in the matrices K_m in (21) are explicitly given, and are implemented in Matlab. We compute the source coefficients by solving

$$\bar{S}_m^i = K_m^{-1} \bar{U}_m^i, \quad \forall m \in I_\Lambda, i \in [-m, \dots, m], \quad (41)$$

where $\bar{U}_m^i$ are the vectors of measurement coefficients calculated from the noisy measurements in (40). Ultimately, the source is expanded in terms of the eigenvectors in S_Λ . The number of points necessary for a good estimate varies with the nature of the measurement and the dimension of the source space.

5.1. Reconstructions

The measurements are generated as outlined above, and we add noise with a noise level $\delta = 0.1$, i.e. 10% noise to all measurements before we perform the reconstruction. We sample the measurements at $N = 201$ and $N = 10001$ points. We plot the $(0, y, z)$ -sections of the sources, the reconstructions and their (log) pointwise absolute difference. Figures 1–3 show the results of the numerical simulations. Figures 1 and 3, made using 10001 sampling points, show that the source s_E agrees remarkably well with the reconstruction from high-frequency measurements $\{U_{k_j}\}_{k_j \in Q_H}$. The outstanding agreement in the presence of noisy measurements might be due to the small number of measurement coefficients being non-zero because of the very simple structure of s_E . From the low-frequency measurements $\{U_{k_j}\}_{k_j \in Q_L}$ we can only reconstruct the projection $P_{S_{\Lambda_L}} s_B$, and hence we miss the component $0.5\varphi_{12,6,7}$ of s_E . In figure 3, we see that the reconstruction of s_B from the low-frequency measurements captures the characteristic functions with larger supports quite well, but that the smallest ball is almost invisible. This is as expected, since S_{Λ_L} does not contain the high-frequency vectors needed to resolve smaller features. On the contrary, the reconstruction of s_B from the high-frequency measurements captures all characteristic functions quite well and is a satisfactory approximation to s_B . As we can see in the difference plots, both the low and high-frequency reconstructions suffer from larger pointwise errors around the discontinuities of s_B , and we notice the well-known Gibbs phenomenon.

The remarkable quality of the reconstructions in figures 1 and 3 is also due to significant oversampling (10001 measurement points) of noise in the measurement and the resulting improvement of the signal-to-noise ratio. To investigate the effect of oversampling on the source reconstruction, we here include figure 2. The results in that figure are obtained using only 201 sampling points, with the same noise level of 10%. While these results are indeed of lower quality than the results of the oversampled case, they still seem quite good.

The results verify that the proposed reconstruction method offers stable reconstruction of sources in $S_\Lambda(B_0)$ and of projections $P_{S_{\Lambda_L}} s$ for $s \in L^2(B_0)$, and that the quality of such reconstructions depends on the number of measurement frequencies. Moreover, the method has a low computational cost, requiring only addition and inversion of low-dimensional matrices.

5.2. Approximation properties of S_{Λ_M}

We now study numerically the approximation properties of S_{Λ_M} . The approximation property we are interested in the distance $\|P_{S_{\Lambda_M}} s - s\|_{L^2(B_0)}$. We want to compare approximations of

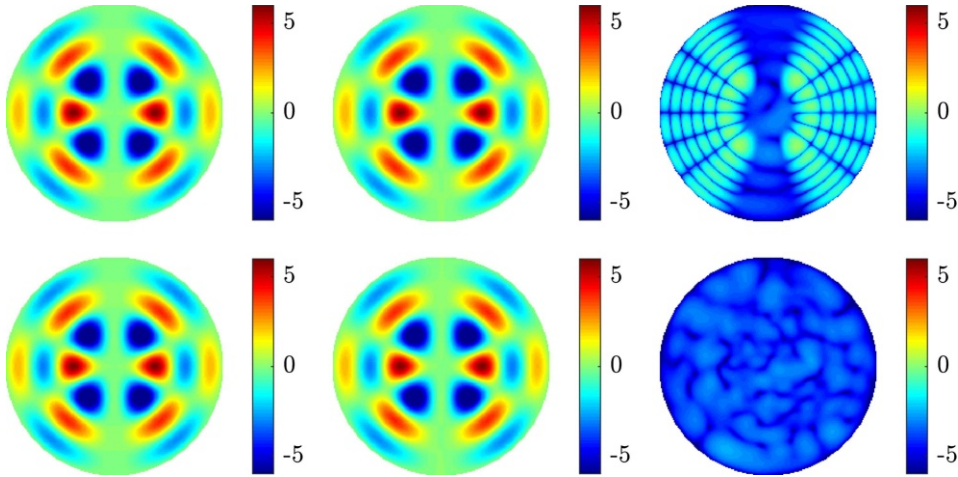


Figure 1. All plots are in the $(0, y, z)$ -plane of B_0 . The reconstruction is made using 10 001 sampling points. Top-left: source $s_E(x)$. Top-middle: Reconstruction $\hat{s}_E^L(x)$ of $s_E(x)$ from low-frequency measurements $\{U_{k_j}\}_{k_j \in Q_L}$. Top-right: Logarithm of pointwise absolute error, $\log |\hat{s}_E^L(x) - s_E(x)|$. Bottom-left: Source $s_E(x)$. Bottom-middle: Reconstruction $\hat{s}_E^H(x)$ of $s_E(x)$ from high-frequency measurements $\{U_{k_j}\}_{k_j \in Q_H}$. Bottom-right: Logarithm of pointwise absolute error, $\log |\hat{s}_E^H(x) - s_E(x)|$.

$s \in L^2(B_0)$ in S_{Λ_M} and in E_Q , where E_Q is a large set of right-singular vectors of the measurement operators F_{k_j} , $k_j \in Q$.

Let $Q = \{k_j\}_{j=1}^N$ be the smallest frequency set associated with Λ_M and $s \in L^2(B_0)$. We define the truncated measurement operator at frequency k_j by

$$F_{k_j}^T = \sum_{m=1}^{\beta_-^{k_j}} \sum_{n=-m}^m \sigma_m^{k_j} v_{m,n}^{k_j}(\cdot, u_{m,n}^{k_j})_{L^2(\Omega)}, \quad k_j \in Q, \quad \beta_-^{k_j} = \arg \min_{m \in \mathbb{N}_0} \{\mu_{m,1} \geq k_j R_0\}.$$

Next, we define the multi-frequency and truncated multi-frequency measurement operators by, respectively,

$$F_Q = \begin{bmatrix} F_{k_1} \\ F_{k_2} \\ \vdots \\ F_{k_N} \end{bmatrix}, \quad F_Q^T = \begin{bmatrix} F_{k_1}^T \\ F_{k_2}^T \\ \vdots \\ F_{k_N}^T \end{bmatrix}. \quad (42)$$

We have that $F_Q, F_Q^T: L^2(B_0) \rightarrow (L^2(\partial B))^N$. Since F_Q^T is finite-rank, it has a finite singular system $\{\sigma_m, \phi_m, \psi_m\}_{m=1}^M$, and the inverse of F_Q^T is

$$(F_Q^T)^{-1} = \sum_{m=1}^M \frac{(\cdot, \psi_m)_{(L^2(\partial B))^N}}{\sigma_m} \phi_m.$$

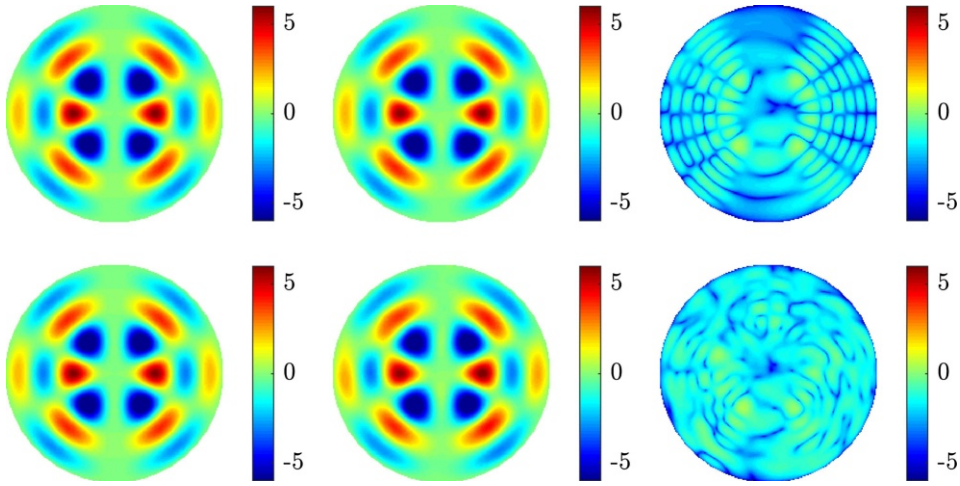


Figure 2. All plots are in the $(0, y, z)$ -plane of B_0 . The reconstruction is made using 201 sampling points. Top-left: Source $s_E(x)$. Top-middle: Reconstruction $\hat{s}_E^L(x)$ of $s_E(x)$ from low-frequency measurements $\{U_{k_j}\}_{k_j \in Q_L}$. Top-right: Logarithm of pointwise absolute error, $\log |\hat{s}_E^L(x) - s_E(x)|$. Bottom-left: Source $s_E(x)$. Bottom-middle: Reconstruction $\hat{s}_E^H(x)$ of $s_E(x)$ from high-frequency measurements $\{U_{k_j}\}_{k_j \in Q_H}$. Bottom-right: Logarithm of pointwise absolute error, $\log |\hat{s}_E^H(x) - s_E(x)|$.

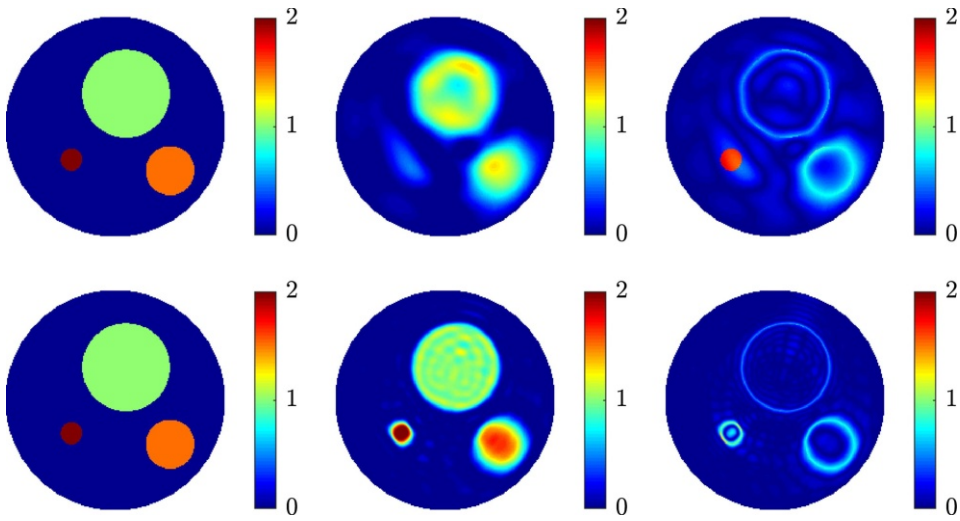


Figure 3. All plots are in the $(0, y, z)$ -plane of B_0 . The reconstruction is made using 10 001 sampling points. Top-left: Source $s_B(x)$. Top-middle: Reconstruction $\hat{s}_B^L(x)$ of $s_B(x)$ from low-frequency measurements $\{U_{k_j}\}_{k_j \in Q_L}$. Top-right: Pointwise absolute error $|\hat{s}_B^L(x) - s_B(x)|$. Bottom-left: Source $s_B(x)$. Bottom-middle: Reconstruction $\hat{s}_B^H(x)$ of $s_B(x)$ from high-frequency measurements $\{U_{k_j}\}_{k_j \in Q_H}$. Bottom-right: Pointwise absolute error $|\hat{s}_B^H(x) - s_B(x)|$.

We say that $(F_Q^T)^{-1}$ is the truncated singular value decomposition (TSVD) inverse of F_Q , and we have that $\text{Im}(F_Q^T)^{-1} = \text{span}\{\phi_m\}$. Since the inversion method using S_{Λ_M} only extracts parts of the recoverable information from the measurements, it is of interest to compare it to approximations in $E_Q = \text{span}\{\phi_m\}$, which in a sense is the largest space of stably recoverable information we can reconstruct from the measurements $\{U_{k_j}\}_{k_j \in Q}$.

To this end, we proceed as follows. For $k_j \in Q$, let E_{k_j} be the set of right-singular vectors included in $F_{k_j}^T$, i.e.

$$E_{k_j} = \{u_{m,n}^{k_j} : 0 \leq m \leq \mathcal{B}_-^{k_j}, \quad -m \leq n \leq m\}. \quad (43)$$

For our investigation, we need the following lemma.

Lemma 5. $\text{Im}(F_Q^T)^{-1} = \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}$.

Proof. First, note that $\text{Ker } F_Q^T = \bigcap_{k_j \in Q} \text{Ker } F_{k_j}^T$, and that $\text{Ker } F_{k_j} = L^2(\Omega) \ominus \text{span } E_{k_j}$. Hence

$$\text{Ker } F_Q^T = \bigcap_{k_j \in Q} (L^2(\Omega) \ominus \text{span } E_{k_j}) = L^2(\Omega) \ominus \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}.$$

The result now follows by noting that

$$\text{Im}(F_Q^T)^{-1} = (\text{Ker } F_Q^T)^\perp = \left(L^2 \ominus \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}\right)^\perp = \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}.$$

□

Hence we have $E_Q = \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}$. To compare approximations, we compute the distances $d(s, S_{\Lambda_M})$ and $d(s, E_Q)$ for different $s \in L^2(B_0)$ and different Λ_M and Q .

Since S_{Λ_M} is the span of orthonormal vectors,

$$d(s, S_{\Lambda_M})^2 = \|s\|^2 - \sum_{\varphi_{\alpha,\beta,\gamma} \in S_{\Lambda_M}} |(s, \varphi_{\alpha,\beta,\gamma})|^2. \quad (44)$$

From chapter 7.6 in [25], we have that the distance from a finite-dimensional subspace $M = \text{span}\{\phi_m\}_{m=1}^M$ of an infinite-dimensional Hilbert space X to any $s \in X$ is given by the formula

$$d(s, M)^2 = \|s\|_X^2 - \beta^T G^{-1} \beta. \quad (45)$$

Here $\beta = [(s, \phi_1), \dots, (s, \phi_M)]$ and G^{-1} is the inverse of the Gram matrix $G = \{(\phi_i, \phi_j)_X\}_{i,j=1}^M$.

Since $E_Q = \text{span}\left\{\bigcup_{k_j \in Q} E_{k_j}\right\}$, the entries of G are given by $(u_{m,n}^{k_i}, u_{\alpha,\beta}^{k_j})_{L^2(\partial B)}$. By an analogous calculation to that in the proof of Lemma 3, we find that

$$\begin{aligned} & (u_{m,n}^{k_i}, u_{\alpha,\beta}^{k_j})_{L^2(B_0)} \\ &= \delta_{m,\alpha} \delta_{n,\beta} \frac{4}{\pi a_m(k_i R_0) a_m(k_j R_0)} \frac{\sqrt{k_i k_j}}{k_i^2 - k_j^2} (k_j j_m(k_i R_0) j_{m-1}(k_j R_0) - k_i j_{m-1}(k_i R_0) j_m(k_j R_0)). \end{aligned}$$

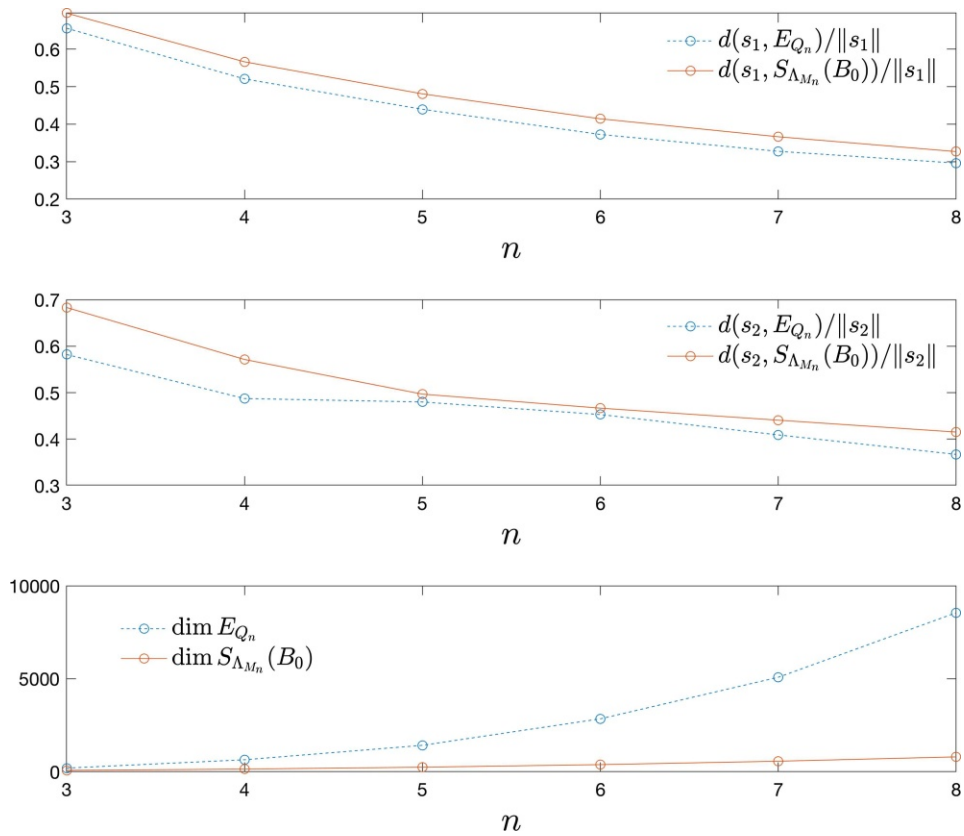


Figure 4. The top and middle plot shows the relative approximation errors in $S_{\Lambda_{M_n}}$ and E_{Q_n} . The bottom plot shows the dimension of the two spaces.

For $n = 3, 4, \dots, 8$, let $\Lambda_{M_n} = \{\lambda_{\alpha,\beta}\}_{\alpha=0,\beta=1}^{n,n}$, and take Q_{M_n} to be the smallest frequency set such that the hypothesis of Theorem 3 holds. We set $R_0 = 1$ and compute the distances from the spaces $S_{\Lambda_{M_n}}$ and E_{Q_n} to the functions

$$s_1(x) = \exp(-|x|^2/0.1) \sin(2 \tan^{-1}(x_2/x_1)) \cos^{-1}(x_3/|x|),$$

$$s_2(x) = \chi_{r,x_0}, \quad r = 0.2, \quad x_0 = (0.2, 0.3, 0.1).$$

The inner products in equations (44) and (45) are approximated with the Matlab quadrature integral3(). Figure 4 shows the (relative) distances to the sources for increasing n and the dimensions of $E_{Q_n} = \text{Im}(F_{Q_n}^T)^{-1}$ and $S_{\Lambda_{M_n}}$. For both sources the approximation in E_{Q_n} is slightly better than in $S_{\Lambda_{M_n}}$, but the dimension, and hence the computational cost associated with the reconstruction/approximation increases dramatically with n . Hence the results indicate that there is a large number of redundant vectors in E_{Q_n} , and that the lower-dimensional space S_{Λ_M} does capture the essential information about the sources present in the multi-frequency measurements.

6. Conclusion

We have analyzed the feasibility and the stability of a solution of the multi-frequency inverse source problem for the Helmholtz equation in $\mathbf{R}^3$. Given any finite set of measurement frequencies, we have characterized certain finite-dimensional source spaces in which exact source reconstruction is guaranteed if the measurements are noise-free. Conversely, for each of the finite-dimensional source spaces we have found the smallest set of measurement frequencies that is sufficient for exact reconstruction in the absence of noise. Regarding stability, we have extended our results from the two-dimensional case and proved a lower bound on the ‘bandwidth’ of the measurement operator, at any measurement frequency. We showed that our inversion method for multi-frequency measurements is stable and highly efficient; since the method does not rely on discretization of the measurement operator, one avoids the computational cost associated with wave phenomena in higher dimensions. It would be interesting to see if the method can be generalized to measurement boundaries other than spheres, as well as to compare its performance against commonly used inversion techniques. Finally, it remains to investigate whether also in the non-constant coefficient case a similar relationship exists between the eigenvalues and eigenvectors of an elliptic PDE and the frequency dependence of the SVD of the measurement operator.

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