

Feynman's Inverse Problem*

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Abstract. We analyze an inverse problem for water waves posed by Richard Feynman in the BBC documentary *Fun to Imagine*. We show that the problem can be modeled as an inverse Cauchy problem for gravity-capillary waves, conduct a detailed analysis of the Cauchy problem, and give a uniqueness proof for the inverse problem. Somewhat surprisingly, this results in a positive answer to Feynman's question. In addition, we derive stability estimates for the inverse problem for both continuous and discrete measurements, propose a simple inversion method, and conduct numerical experiments to verify our results.

Key words. inverse problems, water waves, partial differential equations, mathematical modeling

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1. Introduction. Richard Feynman is one of the most iconic physicists of the 20th century and needs little introduction. In contrast to most earlier giants of science, Feynman was alive after the invention of the video camera, and this resulted in numerous entertaining and inspiring recordings of him talking about science and philosophy. In one such recording, the BBC documentary series *Fun to Imagine* from 1983, there is a segment¹ where Feynman talks about waves:

If I'm sitting next to a swimming pool, and somebody dives in [...], I think of the waves and things that have formed in the water. And when there's lots of people who have dived in the pool there is a very great choppiness of all these waves all over the water. And to think that it's possible, maybe, that in those waves is a clue to what's happening in the pool, that some sort of insect or something, with sufficient cleverness could sit in the corner of the pool and, just be disturbed by the waves, and by the nature of the irregularities and bumping of the waves, have figured out who jumped in where and when, and what's happening all over the pool.

With Feynman's characteristic enthusiasm, we are presented with an inverse problem for water waves, that is, a problem where one attempts to estimate some unknown information ("what's happening all over the pool") from an indirect measurement ("the irregularities and bumping of the waves" observed by the insect), causally related through the physics of water waves.

Inverse problems are a central part of modern science, with applications in, for example, medical imaging, seismology, astronomy, and machine learning. Inverse

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¹<https://www.youtube.com/watch?v=R7zpQLkaAWE>. It is recommended that you watch the clip before you go on reading the paper.

problems are challenging, and they are often called ill-posed problems due to the possible lack of a unique solution and the instability of the inversion procedure. Indeed, it is often impossible to determine the exact cause of a measurement. However, inverse problems for some types of wave propagation, such as electromagnetic and acoustic waves, are among the more tractable examples, with a long list of successful applications [34, 28]. We are, however, dealing with a rather different type of wave, namely, water waves. Again, Feynman in his *Lecture on Physics, Vol. 1* gives the following characterization of water waves:

Now, the next wave of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light: they have all the complications that waves can have.

In this paper we introduce Feynman's inverse problem for water waves. Using a linear two + three dimensional wave model (2D-3D) wave model, we conduct a detailed analysis of the forward and inverse problem. The question we want to address is, "How well does information propagate in water waves?" We show that the answer to this question is "quite well" and explain some of the reasons. First, we give a positive answer to Feynman's question in the form of a uniqueness result (subsection 4.1, Theorem 4.1). The answer is a bit surprising. It says that an observer measuring the wave amplitude and water velocity in any small area of the surface for an arbitrary short time can, in principle, determine the cause of the waves. The result relies on the nonlocality of the system of partial differential equations (PDEs).

In subsection 4.2 we use a spectral observability technique to obtain stability estimates for the inverse problem in the case where we have amplitude measurements of the waves along two adjacent sides of the pool, and in subsection 4.3 we give a similar estimate for discrete measurements. Last, in subsection 5.2 we present a solution method for the inverse problem and test it on simulated data.

Compared to the aforementioned acoustic or electromagnetic waves, inverse problems for water waves seem to have received less attention. However, there are some very interesting earlier works, and we now mention some of these. In [36], the authors use measurements of the surface wave to compute the near-surface currents, and in [40] and [14] the authors aim to reconstruct the bottom topography from similar measurements. The paper [35] surveys several inverse problems related to free surface flows, and the topic of control and observability of both linear and nonlinear water waves has been studied in, e.g., [2, 31, 30].

2. Mathematical Model.

2.1. The Forward Problem. We consider a rectangular pool of uniform depth. The geometry is then determined by the length L , the width W , and the depth H . We assume $W \leq L$. Let $X = (x_1, x_2)$. The domain occupied by the water is

$$\Omega_P = \{(X, z) \in \mathbb{R}^3 : 0 < x_1 < L, 0 < x_2 < W, -H < z < 0\}.$$

Next, we define

$$\begin{aligned}\Gamma_P &= \{(X, z) \in \overline{\Omega}_P \setminus \Omega_P : -H \leq z < 0\}, \\ \Gamma_S &= \{X : 0 < x_1 < L, 0 < x_2 < W\},\end{aligned}$$

where $\overline{\Omega}_P$ denotes the closure of Ω_P and Γ_P and Γ_S represent the walls and bottom of the pool and the flat surface, respectively.

To define the PDE model, we let $\Delta_X = \partial_{x_1}^2 + \partial_{x_2}^2$ and $\Delta_{X,z} = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_z^2$ be the 2D and 3D Laplacians, respectively, and let ∂_ν denote the outward pointing normal derivative on Γ_P, Γ_S , and $\partial\Gamma_S$.

We denote surface wave amplitude by $\eta(X, t)$, i.e., the height of the wave at the point $X \in \Gamma_S$ at time t is $z = \eta(X, t)$. Next, we assume that the water is incompressible and irrotational, so the flow can be described in terms of a velocity potential $\phi(X, z, t)$. The velocity field is given by $V = \nabla_{X,z}\phi$, and the potential satisfies the Laplace equation $\Delta_{X,z}\phi = 0$. Since there is no flow of water through Γ_P , we impose a Neumann boundary condition on the potential,

$$V \cdot \nu = \partial_\nu \phi = 0, \quad (X, z) \in \Gamma_P, t > 0.$$

Figure 1 illustrates the domain and quantities.

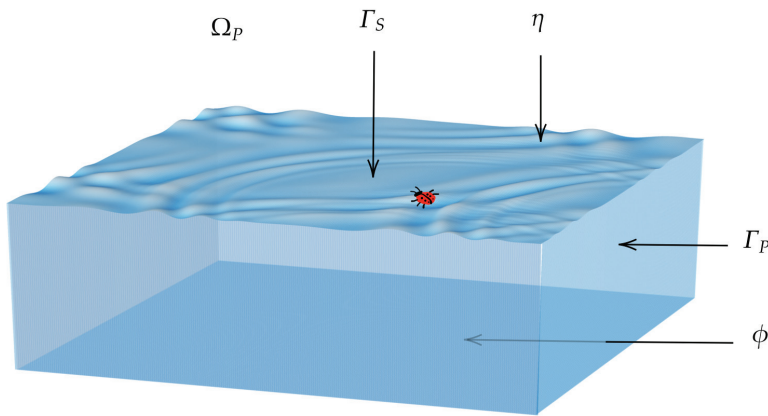


Fig. 1 The domain Ω_P . The surface wave amplitude η is defined on Γ_S , while ϕ is defined in all of Ω_P . The homogeneous Neumann condition is posed on the submerged pool boundary Γ_P . The question is whether it is possible for the ladybug to figure out the initial state of this wave field from its local observation.

We model the disturbance of the initially calm water, the “lots of people who have dived in the pool,” as the initial condition for the surface wave amplitude denoted by η_0 , together with the initial condition for the velocity potential ϕ at the surface, denoted by φ_0 . To model the surface waves, we consider the Airy wave model with surface tension. This is a linearized wave model, and a derivation of it together with a discussion of its validity can be found in, e.g., [1, Chap. 5]. Although one can show that surface tension has little influence on the behavior of longer waves (gravity waves), it is this effect that makes it possible for the insect to sit on the surface. Also, it becomes important when one considers shorter waves (capillary waves), and it turns out to be central to establish stability of the inverse problem.

The small parameter in the linearization is the wave steepness $s = ka$, where k is the magnitude of the wavenumber and a is the maximum amplitude, and the linearization is known to be a good approximation when $s \ll 1$ and $a \ll H$ (cf. [24,

Chap. 1] and references therein). The system of PDEs is

$$(2.1) \quad \left\{ \begin{array}{l} \partial_t \eta - \partial_z \phi|_{z=0} = 0, \quad X \in \Gamma_S, t > 0, \\ \partial_t \phi|_{z=0} + g\eta - S\Delta_X \eta = 0, \quad X \in \Gamma_S, t > 0, \\ \Delta_{X,z} \phi = 0, \quad (X, z) \in \Omega_P, t > 0, \\ \partial_\nu \phi = 0, \quad (X, z) \in \Gamma_P, t > 0, \\ \partial_\nu \eta = 0, \quad X \in \partial\Gamma_S, t > 0, \\ (\eta|_{t=0}, \phi|_{z=0, t=0}) = (\eta_0, \varphi_0), \quad X \in \Gamma_S, \end{array} \right.$$

where g is the gravitational acceleration and $S = \frac{\sigma}{\rho}$, where σ is the surface tension coefficient and ρ is the mass density of water.

The first equation in (2.1) says that the velocity of the surface amplitude should equal the vertical velocity of the water at the surface, while the second equation results from linearizing the dynamic Bernoulli equation at the surface, with the pressure due to surface curvature, i.e., the surface tension, given by $S\Delta_X \eta$. In the bulk of the water, $\Delta_{X,z} \phi = 0$ ensures the incompressibility condition $\nabla_{X,z} \cdot V = 0$. The boundary condition on η ensures that the waves are reflected at the edges of the pool, and the last equation is the initial condition.

2.2. Measurements. We consider several different measurements. Let $\Gamma_M \subset \Gamma_S$ be a subset of the surface, and for $0 \leq t_0 < t_1$, let $I_T = [t_0, t_1]$ be the measurement time interval. In the analysis of the inverse problem for Feynman's insect, we consider the measurement $\mathcal{M}$ to be

$$\mathcal{M}(X, t) = (\eta(X, t), \nabla_X \phi(X, 0, t)), \quad X \in \Gamma_M, \quad t \in I_T.$$

That is, we measure the wave amplitude η and horizontal water velocity $(V_1, V_2) = \nabla_X \phi(X, 0, t)$ on the set Γ_M , i.e., the “irregularities and bumping of the waves” observed by the insect on the surface. In subsections 4.2 and 4.3, we assume that we only measure the amplitude η and consider noisy measurements, measurements of the wave velocity, and discrete measurements.

3. Analysis of the Water Wave Model. Before turning to the inverse problem, we introduce a convenient framework for the analysis and derive some useful properties of (2.1). For related work on linear (and nonlinear) water waves, see, for example, [18, 8, 24, 11, 1].

3.1. Preliminaries. First, note that for conservation of mass to hold in the pool, we must have $\text{vol}(\Omega_P) = \int_{\Gamma_S} (H + \eta(X, t)) dX$ and hence $\int_{\Gamma_S} \eta(X, t) dX = 0$ for all $t \geq 0$. This motivates the use of the function space

$$\dot{L}^2 = \left\{ u \in L^2(\Gamma_S) : \int_{\Gamma_S} u dX = 0 \right\},$$

equipped with the L^2 -inner product. Note that $L^2(\Gamma_S) = \{\text{constant functions}\} \cup \dot{L}^2(\Gamma_S)$, and so $\dot{L}^2$ is a Hilbert space. Next, we use the functions

$$\psi_{m,n}(X) = C_{m,n} \cos(\pi m x_1 / L) \cos(\pi n x_2 / W), \quad C_{m,n} = \begin{cases} \frac{\sqrt{2}}{\sqrt{LW}} & \text{if } m \text{ or } n = 0, \\ \frac{2}{\sqrt{LW}} & \text{otherwise,} \end{cases}$$

with $m, n \in \mathbb{N}_0$. One can verify that $\{\psi_{m,n}\}_{m+n>0}$ is an orthonormal basis for $\dot{L}^2$. We write $\hat{u}_{m,n} = (u, \psi_{m,n})_{L^2}$, so $u \in \dot{L}^2$ has the expansion $u = \sum_{m+n>0} \psi_{m,n} \hat{u}_{m,n}$. We also introduce the Sobolev spaces $\dot{H}^s$,

$$\dot{H}^s = \left\{ u \in \dot{L}^2 : \sum_{m+n>0} (m^2 + n^2)^s |\hat{u}_{m,n}|^2 < \infty \right\}, \quad s \in \mathbb{R},$$

with the inner product $(u, v)_{\dot{H}^s} = \sum_{m+n>0} (m^2 + n^2)^s \hat{u}_{m,n} \hat{v}_{m,n}$.

We can simplify equation (2.1) by considering the trace of the velocity potential at the surface. Let $\varphi(X, t) = \phi(X, z, t)|_{z=0}$. To reduce (2.1) to an equation for (η, φ) , we introduce the Dirichlet-to-Neumann (DN) operator $\mathcal{G}$,

$$(3.1) \quad \mathcal{G}\varphi = \partial_z \phi|_{z=0}, \quad \text{where } \phi \text{ solves } \begin{cases} \Delta_{X,z} \phi = 0 & \text{in } \Omega_P, \\ \partial_\nu \phi = 0 & \text{on } \Gamma_P, \\ \phi|_{z=0} = \varphi & \text{on } \Gamma_S. \end{cases}$$

Furthermore, we write $\mathcal{L} = (S\Delta_X - g)$. We can now rewrite (2.1) as

$$(3.2) \quad \frac{d}{dt} \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \eta \\ \varphi \end{pmatrix}, \quad \begin{pmatrix} \eta \\ \varphi \end{pmatrix}_{t=0} = \begin{pmatrix} \eta_0 \\ \varphi_0 \end{pmatrix}.$$

This is known as the Zakharov–Craig/Sulem formulation (cf. [25]). It reduces the 2D-3D system (2.1) to a 2D-2D system for (η, φ) , at the expense of introducing the nonlocal DN operator. In section 7, we prove some useful properties of $\mathcal{G}$ and $\mathcal{L}$ using their spectral representation in the basis $\{\psi_{m,n}\}$.

3.2. Well-Posedness of Waves in the Pool. We introduce the energy space H_E for the system (3.2). We write

$$A = \begin{bmatrix} 0 & \mathcal{G} \\ \mathcal{L} & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \eta \\ \varphi \end{bmatrix}.$$

We set $H_E = \dot{H}^1 \times \dot{H}^{1/2}$, equipped with the inner product

$$\langle U, V \rangle_E = S(\nabla u_1, \nabla v_1)_{L^2} + g(u_1, v_1)_{L^2} + (\mathcal{G}u_2, v_2)_{L^2}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H_E.$$

It follows from the considerations in section 7 that this is a proper inner product, and that $D(A) = \dot{H}^{5/2} \times \dot{H}^2$ is a suitable domain for A , i.e., $A : D(A) \rightarrow H_E$ is continuous. The H_E -norm is a physical energy norm since the term $S(\nabla \eta, \nabla \eta)_{L^2}$ represents the elastic energy due to the surface tension and the terms $g(\eta, \eta)_{L^2}$ and $(\mathcal{G}\varphi, \varphi)_{L^2} = \int_{\Omega_P} V^2 dX$ are proportional to the potential and kinetic energy of the water, respectively. Next, note that

$$\begin{aligned} \langle U, V \rangle_E &= S(\nabla u_1, \nabla v_1)_{L^2} + g(u_1, v_1)_{L^2} + (\mathcal{G}u_2, v_2)_{L^2} \\ &= -(u_1, \mathcal{L}v_1)_{L^2} + (u_2, \mathcal{G}v_2)_{L^2}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(A). \end{aligned}$$

Since $\dot{U} = AU$, we see that $\|U\|_E^2 = \langle U, U \rangle_E^2$ is conserved. Assuming U solves (3.2), we have

$$\frac{d}{dt} \left(\frac{1}{2} \|U\|_E^2 \right) = \langle \dot{U}, U \rangle_E = -(\mathcal{G}u_2, \mathcal{L}u_1)_{L^2} + (\mathcal{L}u_1, \mathcal{G}u_2)_{L^2} = 0.$$

We are now in a position to show well-posedness of (3.2). The proof, which follows the semigroup approach used in [30], is found in section 7.

THEOREM 3.1. *For each $U_0 = (\eta_0, \varphi_0) \in H_E$, the water wave system (3.2) has a unique solution $U = (\eta, \varphi)^\top \in C([0, \infty); H_E)$ that satisfies $\|U(t)\|_E = \|U_0\|_E$ for all $t \geq 0$. In addition,*

$$(3.3) \quad U(t) = \sum_{m,n \geq 0} \cos(\omega_{m,n}t) \psi_{m,n} \begin{bmatrix} (\eta_0, \psi_{m,n})_{L^2} \\ (\varphi_0, \psi_{m,n})_{L^2} \end{bmatrix} + \sin(\omega_{m,n}t) \psi_{m,n} \begin{bmatrix} \frac{\omega_{m,n}}{g + Sk_{m,n}^2} (\varphi_0, \psi_{m,n})_{L^2} \\ -\frac{g + Sk_{m,n}^2}{\omega_{m,n}} (\eta_0, \psi_{m,n})_{L^2} \end{bmatrix},$$

where

$$(3.4) \quad \omega_{m,n} = \sqrt{(gk_{m,n} + Sk_{m,n}^3) \tanh(k_{m,n}H)} \quad \text{with} \quad k_{m,n} = \pi \sqrt{(m/L)^2 + (n/W)^2}$$

is the dispersion relation for gravity-capillary waves. Moreover, for $(\eta_0, \varphi_0) \in \dot{H}^{s+\frac{1}{2}} \times \dot{H}^s$ with $s \geq 1/2$, we have that

$$\eta \in \dot{H}^{s+\frac{1}{2}} \quad \text{and} \quad \varphi \in \dot{H}^s \quad \text{for all } t \geq 0.$$

Note: To ease notation later, the sum above includes $(m, n) = (0, 0)$, although the term is always zero since $(\psi_{0,0}, f)_{L^2} = 0$ for $f \in \dot{L}^2$.

With the well-posedness of the forward problems established, we are ready to tackle the inverse problem.

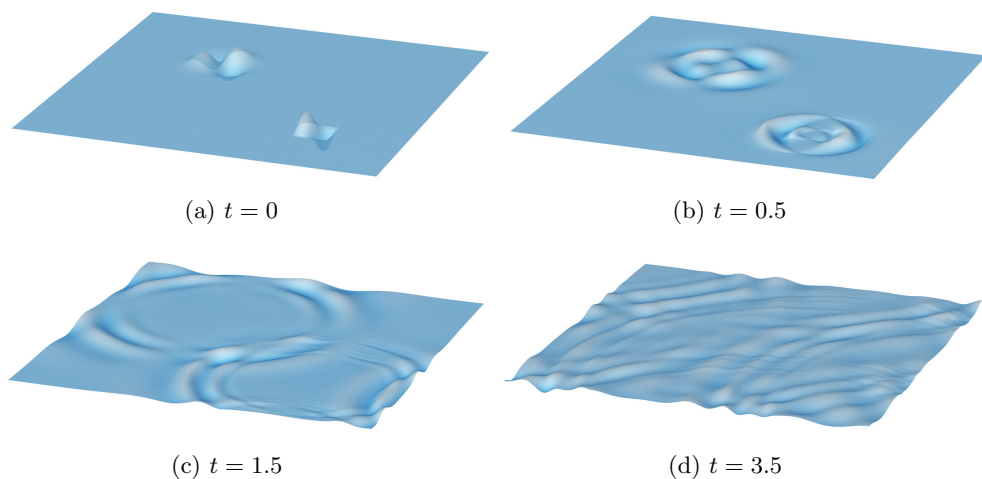


Fig. 2 Four snapshots of the wave amplitude η , with initial conditions and parameters as given in subsection 5.2.

4. The Inverse Problem. We now analyze the inverse problem for several different measurements. We consider the following problems:

- 1: Determination of the initial disturbance from measurement on a small domain.
- 2: Stability of reconstructing the initial disturbance from measurements along the sides of the pool.

- 3: Stability of reconstruction of the initial disturbance from discrete measurements.

4.1. Problem 1: Determination of the Initial Disturbance for Feynman's Clever Insect. We assume the measurement domain Γ_M is a nonempty, open subset of the surface Γ_S , and let $I_T = [t_0, t_1]$ be the measurement time interval. Let the measurement be $\mathcal{M}(X, t) = (\eta(X, t), \nabla_X \varphi(X, t))$ with $(X, t) \in \Gamma_M \times I_T$.

THEOREM 4.1. *Assume $(\eta_0, \varphi_0) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^1$. Then the measurement $\mathcal{M}(X, t)$ uniquely determines the initial disturbance (η_0, φ_0) .*

For waves in the linear regime, the answer to Feynman's question is therefore positive.

COROLLARY 4.2. *A sufficiently clever insect can determine the initial disturbance in the pool from the measurement $\mathcal{M}$.*

Remark 4.3. The above result might seem strange when considering the limitations to inverse problems for other types of waves, e.g., acoustic and electromagnetic waves. For the two latter types of waves, information propagates at a finite speed. Hence, for any finite measurement time, an initially localized disturbance will not even be detectable by an observer positioned sufficiently far away.

Mathematically, this is seen by considering the Cauchy problem for the wave equation in $\mathbb{R}^n$,

$$(\partial_t^2 - c^2 \Delta)u = 0, \quad (u(0), \partial_t u(0)) = (u_0, u_1).$$

This equation describes the propagation of acoustic and electromagnetic waves in a homogeneous medium with wave speed c . Huygens' principle says that if $u_0(X) = u_1(X) = 0$ outside some ball $B_R(0)$ with radius R , then $u(t, X) = 0$ outside of the ball B_{R+ct} for all $t \geq 0$ (cf. [12, Chap. 6]).

The result also has the strange consequence that if the water is still on some open set on the surface for a tiny duration of time, then the whole body of water must have been still everywhere and for all time.

As becomes clear in the proof, it is the nonlocal behavior of the DN operator that is the mechanism enabling this local uniqueness property; due to the modeling of the fluid as incompressible and irrotational, the velocity potential is harmonic. This allows for the application of the unique continuation property, but it is still surprising that this property propagates enough information to ensure the unique determination of the initial disturbance. For a physical intuition of this effect, one can think of stepping into a bathtub; as one descends into the water, the water level appears to increase by the same amount everywhere and at once. Apparently, the local disturbance manifests itself globally and instantly.

We note that the unique continuation property has been established for various water wave models before (cf. [20, 21, 42]) and for the Schrödinger equation, which is also a dispersive PDE [38]. Moreover, there has recently been much interest in inverse problems related to fractional PDEs [32, 19]. Fractional differential operators are nonlocal operators, and for some of the inverse problems considered for such operators, unique continuation properties and local uniqueness results can be established (cf. [10, 16, 15]).

Proof of Theorem 4.1. The proof relies on the energy estimate in Theorem 3.1 and the unique continuation property from Cauchy data for elliptic PDEs, and we paraphrase here Corollary 2.22 in [9]: *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume*

$\Delta u = 0$ in Ω . Let Γ be a nonempty open subset of $\partial\Omega$. Then $u|_{\Gamma} = \partial_{\nu}u|_{\Gamma} = 0$ implies that $u = 0$ in Ω .

By Theorem 3.1, the measurement is well-defined in the sense that $\int_{\Gamma_S} \mathcal{M} h dX$ is bounded for all $h \in C^\infty(\Gamma_S)$. Now, assume $\mathcal{M}(X, t) = 0$ for $(X, t) \in \Gamma_M \times I_T$. Then $\dot{\eta} = 0$ in some space-time cylinder $Q_\varepsilon = B_\varepsilon \times I_\varepsilon \subset \Gamma_M \times I_T$, and by (3.2) this implies that $\mathcal{G}\varphi = 0$ and $\dot{\varphi} = 0$ in Q_ε . Moreover, since $\nabla_X \varphi = 0$ in Q_ε , we must have $\varphi|_{Q_\varepsilon} = C_\varepsilon$ for some constant C_ε . But this implies $\phi = C_\varepsilon$. Assume there is some $\tilde{\phi} \neq C_\varepsilon$, and such that $\tilde{\phi}|_{Q_\varepsilon} = C_\varepsilon$ and $\partial_z \tilde{\phi}|_{Q_\varepsilon} = 0$. Then $\tilde{\phi} - C_\varepsilon$ is harmonic in Ω_P and satisfies $(\tilde{\phi} - C_\varepsilon)|_{Q_\varepsilon} = \partial_z(\tilde{\phi} - C_\varepsilon)|_{Q_\varepsilon} = 0$. We can now apply the unique continuation result and conclude that $\tilde{\phi} = C_\varepsilon$ in Ω_P . Therefore $\varphi = C_\varepsilon$, and since $\varphi \in \dot{L}^2$, $\varphi = 0$. Hence, with $\dot{\varphi} = 0$, we have that $(S\Delta_X - g)\eta = 0$ on Γ_S . But since $g/S > 0$, the only solution to $-\Delta_X \eta = -\frac{g}{S}\eta$ on Γ_S with homogeneous Neumann conditions is $\eta = 0$. By Theorem 3.1 we have $\|(\eta(X, t), \varphi(X, t))\|_E = \|(\eta_0, \varphi_0)\|_E$ for all $t \geq 0$ and hence $(\eta_0, \varphi_0) = 0$, and since the map $(\eta_0, \varphi_0) \mapsto \mathcal{M}$ is linear, the conclusion now follows. $\square$

Theorem 4.1 says nothing about the stability of the procedure of inversion. In fact, unique continuation is known to be a severely ill-posed problem [13]. Hence, an observer who makes imperfect measurements would most likely not succeed with the task of computing the initial conditions. In the next section, we investigate the stability properties of the inverse problem.

4.2. Problem 2: Uniqueness and Stability from Measurements along Two Adjacent Sides of the Pool. From the study of the observability of wave equations, it is well known that the part of the boundary where one measures the wave should satisfy certain geometric conditions to yield stability and uniqueness [7, 39]. Here, we utilize the explicit solution in Theorem 3.1 and a so-called spectral observability method used in [38, 22, 39]. Assuming that we measure the wave amplitude along two adjacent sides of the pool, i.e.,

$$\mathcal{M}(X, t) = \eta(X, t)|_{X \in \Gamma_M}, \quad \text{with } \Gamma_M = \{(x_1, 0) : x_1 \in (0, L)\} \cup \{(0, x_2) : x_2 \in (0, W)\},$$

we obtain the following result.

THEOREM 4.4. *Assume the measurement time satisfies*

$$(4.1) \quad T \geq \frac{6L\sqrt{L^2 + W^2}}{W} \left(\frac{Hg}{2\sqrt{1 + \frac{H^2g}{3S}}} - \frac{3S}{2H} \right)^{-1/2}.$$

Then there is a constant C , independent of (η_0, φ_0) , such that following stability estimate holds:

$$(4.2) \quad \int_0^T \int_{\Gamma_M} |\mathcal{M}(X, t)|^2 dX dt \geq CT \left(\|\eta_0\|_{L^2}^2 + \|\varphi_0\|_{H^{-\frac{1}{2}}}^2 \right).$$

Remark 4.5. The term

$$\left(\frac{Hg}{2\sqrt{1 + \frac{H^2g}{3S}}} - \frac{3S}{2H} \right)^{-1/2}$$

is a lower bound on the group velocity $c_g = \partial_k \omega(k)$ of the water waves. The group velocity is the velocity with which a localized wave group/wavelet propagates, and it

is also the velocity of energy transport (cf. [1]). The lower bound on the observation time T is given by a term proportional to the maximal length of the pool times the minimal group velocity. This is comparable to similar bounds for acoustic or electromagnetic waves, where the bound is proportional to the minimal wave velocity times the maximal distance (cf. [3]). Moreover, as we see in the proof of Theorem 4.4, the presence of surface tension and the resulting capillary waves is necessary to have stability. In fact, for pure gravity waves ($S = 0$ in (3.4)), the group velocity $c_g(k) = \partial_k \omega(k) \sim k^{-1/2}$ approaches zero as the wavenumber k increases. Consequently, one can have a compactly supported disturbance comprised of modes with frequencies so high that they do not reach the boundary during the measurement time $T \geq 0$. In contrast, this cannot happen for gravity-capillary waves, as $c_g(k) \sim k^{1/2}$.

Equation (4.4) implies uniqueness and stability. If we have a noisy measurement $\tilde{\mathcal{M}} = \mathcal{M} + \varepsilon$ that we attribute to some initial data $(\tilde{\eta}_0, \tilde{\varphi}_0)$, it shows that the difference between $(\tilde{\eta}_0, \tilde{\varphi}_0)$ and the true initial data (η_0, φ_0) is bounded by ε , i.e.,

$$\int_0^T \int_{\Gamma_M} |\varepsilon|^2 dX dt \geq C \frac{T}{L\pi} \left(\|\eta_0 - \tilde{\eta}_0\|_{L^2}^2 + \|\varphi_0 - \tilde{\varphi}_0\|_{\dot{H}^{-\frac{1}{2}}}^2 \right).$$

Additional Stability. The more precise measurements one is able to make, the more accurate the reconstruction should be. For example, if we can measure the amplitude with such precision that we can compute the (vertical) wave velocity without too much error, or if we are able to measure the velocity or acceleration directly, this information should result in a better reconstruction. We can quantify this by expressing the stability estimate with norms that include derivatives. As a simple consequence of Theorem 4.4 we get the following corollary.

COROLLARY 4.6. *Assume $k \in \mathbb{N}_0$, $(\eta_0, \varphi_0) \in \dot{H}^{\frac{3}{2}k} \times \dot{H}^{\frac{3}{2}k - \frac{1}{2}}$, and that T satisfies the same bound as in Theorem 4.4. Then there is a constant C_k , independent of (η_0, φ_0) , such that the following stability estimate holds:*

$$(4.3) \quad \int_0^T \int_{\Gamma_M} |\partial_t^k \eta(X, t)|^2 dX dt \geq C_k T (\|\eta_0\|_{\dot{H}^{\frac{3}{2}k}}^2 + \|\varphi_0\|_{\dot{H}^{\frac{3}{2}k - \frac{1}{2}}}^2).$$

Proofs of the above results are found in section 7 and are based on an analysis of the dispersion relation and application of nonharmonic Fourier analysis.

4.3. Problem 3: Discrete Measurements and Bandlimited Reconstructions.

We now investigate the more practical question of reconstruction from discrete, noisy boundary measurements. By discrete measurements, we mean that the wave amplitude is measured at a finite number of points in time and space, in contrast to the continuum case considered in Theorems 4.1 and 4.4. This is always the case in any real-world experiment or application, and was recently studied in, e.g., [4, 17]. The experimental setup we consider could be a water tank with transparent walls and a camera recording the waves as they hit the wall. The resulting measurement has the form

$$\{\eta(x_i, 0, t_j) : 0 \leq i \leq N, 0 \leq j \leq M\}$$

for some finite $N, M \in \mathbb{N}$. Even in the absence of noise from such measurements, one can in general not obtain reconstructions of an arbitrary initial disturbance. We instead aim to reconstruct finite-dimensional disturbances and introduce the space

$$\dot{H}_\beta = \{u \in \dot{L}^2 : u \in \text{span}\{\psi_{m,n} : 0 \leq m, n \leq \beta\}\}.$$

The space $\dot{H}_\beta$ is a space of bandlimited functions, i.e., functions that are finite linear combinations of the basis functions $\psi_{m,n}$, and where the maximal spatial frequency is given by the bandwidth β . For $f \in \dot{L}^2$, we denote by $P_\beta f$ the projection of f onto $\dot{H}_\beta$, i.e.,

$$P_\beta f = \sum_{m,n \leq \beta} (f, \psi_{m,n})_{L^2} \psi_{m,n}(X).$$

Consider now a measurement along one side of the pool, $\Gamma_M = \{(x_1, 0) : 0 \leq x_1 \leq L\}$, and assume $\eta_0 \in \dot{H}^s$, $s \geq 1$, and $\varphi_0 = 0$. Given $\beta > 0$, we sample η as follows: Let $N_x = \beta + 1$ and set $x_k = (k + 1/2) \frac{L}{N_x}$. For some T and N_t to be determined, set $\Delta t = T/N_t$ and $t_j = j\Delta t$. We introduce two different measurements:

$$(4.4) \quad \begin{aligned} \mathcal{M} &= [\eta(x_k, 0, t_j)] + \varepsilon_{k,j}, \\ \mathcal{M}_\beta &= [P_\beta \eta(x_k, 0, t_j)], \end{aligned} \quad 0 \leq k \leq N_x - 1, \quad 0 \leq j \leq N_t.$$

Here $\mathcal{M}$ is the noisy measurement of the true wave, while $\mathcal{M}_\beta$ is the exact measurement of η with initial condition $P_\beta \eta_0 \in \dot{H}_\beta$, i.e., with a finite-dimensional initial condition. The noise is modeled by the terms $\varepsilon_{j,k}$ which are specified in the numerical experiment section. We use the Frobenius norm $\|\mathcal{M}\|_F^2 = \sum_{k,j} |m_{k,j}|^2$ as a norm on the measurement, and we denote by $R_\beta \eta_0$ the reconstruction of η_0 from the measurement $\mathcal{M}$.

Given a bandwidth β , the following theorem shows how to choose T and N_t such that we can stably approximate $P_\beta \eta_0$ from the measurements $\mathcal{M}$. The proof, found in section 7, is constructive and leads to a simple reconstruction method.

THEOREM 4.7. *Choose $\beta \in \mathbb{N}$, and let*

$$\delta = \frac{4\pi}{5\omega_{\beta,\beta}} \min \left(1/5, \frac{\pi}{L} \left(\sqrt{\beta^2 + 1} - \beta \right) \sqrt{\frac{Hg}{2\sqrt{1 + \frac{H^2g}{3S}}} - \frac{3S}{2H}} \right).$$

Take $N_t \in \mathbb{N}$ such that

$$N_t > \max \left(2\beta + 2, \frac{1}{\delta} \right) \quad \text{and} \quad T = \frac{4\pi N_t}{5\omega_{\beta,\beta}}.$$

Let $\mathcal{M}$ and $\mathcal{M}_\beta$ be as described in (4.4), and assume $\eta_0 \in \dot{H}^s$, $s \geq 1$. Then the following estimates hold:

$$\begin{aligned} \|P_\beta \eta_0 - R_\beta \eta_0\|_{L^2} &\leq \sqrt{\frac{LW}{2N_x \left(N_t - \frac{1}{\delta}\right)}} \|\mathcal{M}_\beta - \mathcal{M}\|_F, \\ \|\eta_0 - R_\beta \eta_0\|_{L^2} &\leq \frac{2}{\beta^s} \|\eta_0\|_{\dot{H}^s} + \sqrt{\frac{LW}{2N_x \left(N_t - \frac{1}{\delta}\right)}} \|\mathcal{M}_\beta - \mathcal{M}\|_F. \end{aligned}$$

Remark 4.8. For a given bandwidth β , the above result quantifies the reconstruction error in terms of how much the true measurement $\mathcal{M}$ deviates from the ideal measurement $\mathcal{M}_\beta$ and in terms of how well η_0 is approximated by $P_\beta \eta_0$. Moreover, one can infer that as the bandwidth β increases, longer observation time and higher sampling frequency are needed for the stability estimate to hold. This makes sense,

since as β increases, we need to “filter out” frequencies that are closer together. However, as we mention in subsection 5.2, the estimates one gets from the above theorem on N_t and T are pessimistic, and it turns out that one also can get good reconstructions from using fewer samples.

5. Inversion: Method and Numerical Experiments. We now propose a reconstruction method for the measurement setup in Theorem 4.7 and test it on synthetic data.

5.1. Numerical Solution of the Forward Problem. To generate synthetic data, we compute a high precision approximation to $\mathcal{M}$ from the solution formula (3.3) and add noise to the measurement.

- 1: Choose the number N_a of eigenfunctions included in the solution (3.3), so that $N_a \gg \beta$, where β is the bandwidth of the reconstruction.
- 2: For initial data η_0 , compute the coefficients $q_{m,n} = (\eta_0, \psi_{m,n})_{L^2}$ using a suitable numerical quadrature method.
- 3: Given the sampling points (x_k, t_j) , let

$$\mathcal{M}(k, j) = \sum_{m, n \leq N_a} \cos(\omega_{m,n} t_j) \psi_{m,n}(x_k, 0) q_{m,n}, \quad 0 \leq k \leq N_x - 1, \quad 0 \leq j \leq N_t.$$

- 4: Add noise. Let N_ε be the relative noise level, i.e.,

$$\frac{\|\varepsilon\|_F}{\|\mathcal{M}\|_F} \leq N_\varepsilon.$$

We assume that the noise is $\varepsilon \sim \mathcal{N}(0, I_{N_x N_t})$, i.e., independent and identically distributed Gaussian random variables. Given a realization of ε' of the noise, we make sure the relative error is achieved by setting $\varepsilon = \varepsilon' N_\varepsilon \|\mathcal{M}\|_F / \|\varepsilon'\|_F$ and take the noisy measurement to be $\mathcal{M}^\varepsilon = \mathcal{M} + \varepsilon$.

5.2. Inversion Method and Numerical Experiments. Based on the proof of Theorem 4.7, we propose the following inversion method. It uses the discrete cosine matrix

$$D_3 = [\cos(\pi(k+1/2)m/N_x)], \quad 0 \leq k \leq N_x - 1, \quad 0 \leq m \leq N_x - 1,$$

and the nonharmonic Vandermonde matrix

$$V_m(j, :) = [(\overline{v_\beta^m})^j \quad \cdots \quad (\overline{v_0^m})^j \quad (v_0^m)^j \quad \cdots \quad (v_\beta^m)^j] \quad \text{for } 0 \leq j \leq N_t,$$

where $V_m(j, :)$ denotes the j th row of the matrix V_m , $v_n^m = e^{i\omega_{m,n}\Delta t}$, $\Delta t = T/N_t$, and β is the bandwidth. We also denote by $\mathcal{C}$ the matrix of the Fourier coefficients of $P_\beta \eta_0$ (should you want to implement the method yourself, we suggest you skim through the proof of Theorem 4.7 for more details).

Reconstruction Method (Nonharmonic Fourier Inversion).

- 1: Choose a bandwidth β . Given the problem parameters L, W, H, g , and S , compute the minimal frequency gap δ and choose the sampling parameters N_x, N_t , and T according to Theorem 4.7.
- 2: Obtain the measurement $\mathcal{M}^\varepsilon$, i.e., the matrix of wave amplitude samples.
- 3: Compute $U = D_3^{-1} \mathcal{M}$. For $m = 0, 1, 2, \dots, \beta$, solve the linear system

$$\mathcal{C}^\top(m, :) = (V_m^\dagger V_m)^{-1} V_m^\dagger U(m, :)^T,$$

where $V_m^\dagger$ denotes the Hermitian transpose of V_m .

- 4: Extract coefficients $\{p_{m,n}\}$ from $\mathcal{C}$ and set $R_\beta \eta_0 = \sum_{m,n \leq \beta} p_{m,n} \psi_{m,n}(X)$.

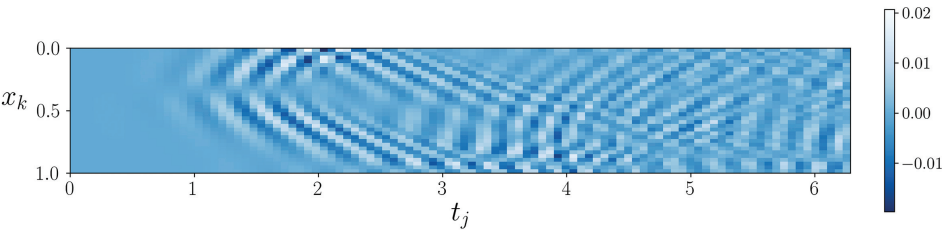


Fig. 3 Plot of the sampled data in $\mathcal{M}$. The pixel $\mathcal{M}(k, j)$ is the measurement value at position x_k and time t_j .

Numerical Experiments. Table 1 summarizes the physical parameters used in the experiments. For the initial condition we choose

$$\begin{aligned} \eta_0(X) = & -\frac{0.1}{\sigma^2}(x_1 - x_1^g)(x_2 - x_2^g) \exp\left(-\frac{(x_1 - x_1^g)^2 + (x_2 - x_1^g)^2}{\sigma^2}\right) \\ & + 0.05\chi_{x_1^s, x_2^s, I} \sin(\pi(x_1 - (x_1^s - I/2))/I) \sin(2\pi(x_2 - (x_1^s/2 - I/2)/I)) - C_0, \end{aligned}$$

where $\sigma = 0.05$, $(x_1^g, x_2^g) = (L/3, W/3)$, $(x_1^s, x_2^s) = (3L/4, 3W/4)$, $I = L/10$, $\chi_{x_1^s, x_2^s, I}$ is the indicator function for the square with side length I centered at (x_1^s, x_2^s) , and C_0 is a constant such that $\int_{\Omega} \eta_0 dX = 0$.

Table 1

g	S	L	W	H
9.81m/s ²	7.08×10^{-5} m ³ /s ²	1m	1m	0.5m

We take the relative noise level to be $N_{\varepsilon} = 0.1$ (i.e., 10%). Furthermore, we take three different values of the bandwidth parameter β and conduct the inversion for each one. The sampling parameters for the different values of β are shown in Table 2.

Table 2

β	N_t	T	Δt	N_x
8	770	103 s	0.134 s	9
16	2979	278 s	0.093 s	17
32	11090	697 s	0.062 s	33

Here we see that the number of samples in time N_t and the sampling time T increases substantially as a function of β . As the conditions in Theorem 4.7 are only sufficient, we in addition carry out the inversion for each given β with $N_t/2$ samples (but Δt as in Table 2).

Figure 2 shows the initial condition and some snapshots of the resulting wave produced by the method outlined in subsection 5.1, while Figure 3 shows a section of a measurement $\mathcal{M}$.

5.3. Results. Tables 3 to 5 give the relevant measures of error in the reconstructions $R_{\beta}\eta_0$, and in Figure 4 we show the reconstructions $R_{\beta}\eta_0$ and the pointwise error. We can see that the quality of the reconstruction, both pointwise and in the

Table 3

β	$\frac{\ \eta_0 - R_\beta \eta_0\ _{L^2}}{\ \eta_0\ _{L^2}}$	$\frac{\ P_\beta \eta_0 - R_\beta \eta_0\ _{L^2}}{\ P_\beta \eta_0\ _{L^2}}$	$\frac{\ \eta_0 - P_\beta \eta_0\ _{L^2}}{\ \eta_0\ _{L^2}}$
8	0.9329	0.0989	0.9322
16	0.5769	0.0177	0.5767
32	0.111	0.008	0.110

Table 4

β	$\ P_\beta \eta_0 - R_\beta \eta_0\ _{L^2}$	$\sqrt{\frac{LW}{2(N_t-1/\delta)N_x}} \ \mathcal{M}_\beta - \mathcal{M}\ _F$
8	0.00014	0.0227
16	0.00005	0.0270
32	0.000033	0.0107

Table 5

β	$\ \eta_0 - R_\beta \eta_0\ _{L^2}$	$\frac{\sqrt{2}}{\beta} \ \eta_0\ _{\dot{H}^1} + \sqrt{\frac{LW}{2(N_t-1/\delta)N_x}} \ \mathcal{M}_\beta - \mathcal{M}\ _F$
8	0.0037	0.0709
16	0.0023	0.0433
32	0.0005	0.0218

L^2 -norm, goes from quite poor to quite good as we increase β . Moreover, we see that the inequalities in Theorem 4.7 hold true, but that they are not very sharp. In a real experiment, the lack of any dissipation mechanism in (2.1) makes the model unrealistic, as there would probably be very few waves bouncing around in a small wave tank after the 10-minute measurement time required for Theorem 4.7 to hold. However, the relative errors on inversion with $T/2$ (as compared to the values in Table 2) are almost identical to those with T , indicating the possibility for improving the bounds on the sampling parameters.

6. Conclusion. In this paper we have analyzed our proposed model of Feynman’s inverse problem. The main result is the unique determination of the initial disturbance from measurement on an open set, giving a positive answer to Feynman’s question. We have also shown that due to the presence of surface tension, one can obtain stability results similar to those that hold for acoustic and electromagnetic waves, from boundary measurements of water waves. Moreover, we have proposed and numerically tested a reconstruction method applicable to discrete, noisy boundary measurements.

As water waves models are mathematically very rich (cf. [25, 1]), there are many more complicated, nonlinear models that can be used to learn more about the propagation of information by water waves (or surface waves, in general). Also, one could consider compressible fluid models or a situation where the disturbance is time-dependent (e.g., moving around). Moreover, the analysis shows that water waves contain a lot of information about their previous state, and so it might be possible to use a similar approach to study conditions leading to rogue waves or other rare surface wave phenomena.

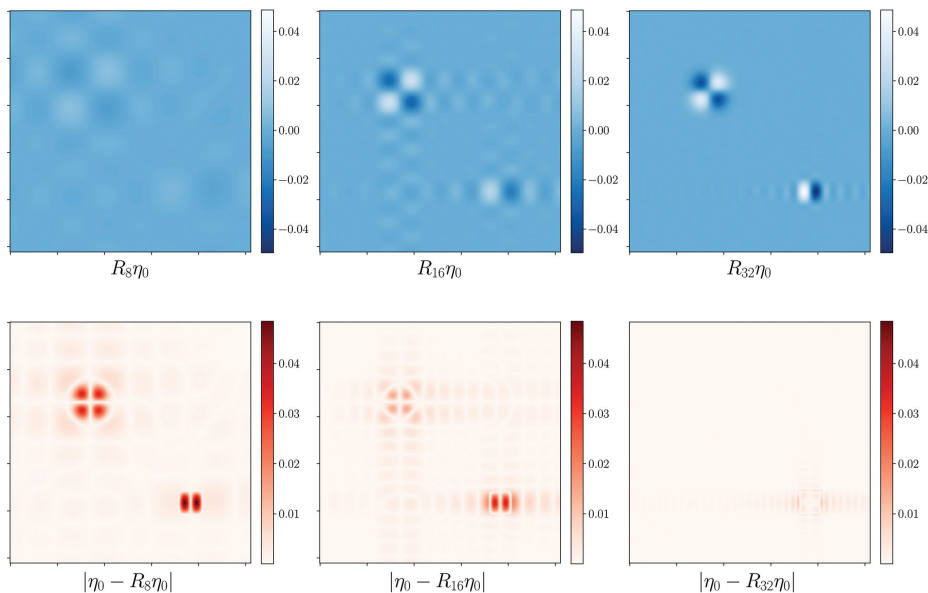


Fig. 4 Plot of the reconstruction $R_{\beta}\eta_0$ and pointwise error $|R_{\beta}\eta_0 - \eta_0|$ for various β . The results are in line with the estimates in Theorem 4.7, and the quality of the reconstruction increases substantially with each doubling of β .

We conclude by going back to the BBC recording. Feynman continues:

And that's what we're doing when we're looking at something. The light that comes out is waves, just like in the swimming pool, except in three dimensions instead of the two dimensions of the pool [...] And we have an eighth of an inch black hole, into which these things go [...] It's quite wonderful that we figure out so easy... that's really because the light waves are easier than the waves in the water, [they are] a little bit more complicated. It would have been harder for the bug than for us, but it's the same idea: to figure out what we are looking at a distance.

We can reasonably say, at least for waves in a truly incompressible fluid in the linear regime, that Feynman was not entirely correct in his argument: It might be harder for the insect to figure out what is happening in the pool, but it is, in theory, possible. And had the insect, in contrast, been observing electromagnetic waves like the eye does (or had the water waves behaved like the electromagnetic waves), this would not be true.

7. Proofs. This section contains the proofs that were not in the main sections, as well as short introductions to the techniques used.

7.1. Proof of Theorem 3.1.

The Operators $\mathcal{G}$ and $\mathcal{L}$. We first consider the DN operator as defined in (3.1). It differs slightly from the standard definition, as it is usually defined on the whole boundary (cf. [28, Chap. 12]). Due to the simple geometry of the pool, we apply

separation of variables to find that the unique solution ϕ in (3.1) is

$$(7.1) \quad \phi(X, z) = \sum_{m+n>0} \psi_{m,n}(X) \frac{\cosh(k_{m,n}(z+H))}{\sinh(k_{m,n}H)} \tanh(k_{m,n}H) (\varphi, \psi_{m,n})_{L^2(\Gamma_S)},$$

where $k_{m,n} = \pi \sqrt{(m/L)^2 + (n/W)^2}$ and $\psi_{m,n}$ are the basis functions defined in subsection 3.1. Hence

$$(7.2) \quad \mathcal{G}\varphi = \frac{\partial}{\partial z} \phi|_{z=0} = \sum_{m+n>0} k_{m,n} \tanh(k_{m,n}H) (\varphi, \psi_{m,n})_{L^2(\Gamma_S)} \psi_{m,n}(X),$$

and we have expressed $\mathcal{G}$ in terms of the Fourier multiplier $|D| \tanh(|D|H)$. Since we can find constants $a_1, a_2 > 0$ such that

$$a_1 |m^2 + n^2|^{1/2} \leq k_{m,n} \tanh(k_{m,n}H) \leq a_2 |m^2 + n^2|^{1/2},$$

we infer that $\mathcal{G} : \dot{H}^s \rightarrow \dot{H}^{s-1}$. Moreover, $\mathcal{G}$ is symmetric on $\dot{H}^s$, and $(\mathcal{G}u, v)_{L^2}$ is an inner product on $\dot{H}^{1/2}$. Also, $\mathcal{G}$ is one-to-one, and since for any $v \in \dot{H}^{s-1}$, we may set

$$u = \sum_{m+n>0} \frac{\hat{v}_{m,n}}{k_{m,n} \tanh(k_{m,n}H)} \psi_{m,n}$$

and compute

$$\|u\|_{\dot{H}^s}^2 = \sum_{m+n>0} \frac{(m^2 + n^2)^s |\hat{v}_{m,n}|^2}{(k_{m,n} \tanh(k_{m,n}H))^2} \leq C \sum_{m+n>0} (m^2 + n^2)^{s-1} |\hat{v}_{m,n}|^2 = C \|v\|_{\dot{H}^{s-1}}^2,$$

it follows that $\mathcal{G} : \dot{H}^s \rightarrow \dot{H}^{s-1}$ is onto, and so it is bijective.

Furthermore, $\mathcal{G} : D(\mathcal{G}) \rightarrow \dot{H}^{1/2}$, $D(\mathcal{G}) = \dot{H}^{3/2}$ is self-adjoint. This follows from the fact that for any linear operator T on $\dot{H}^{1/2}$ that is one-to-one and has a dense range, it holds that $(T^*)^{-1} = (T^{-1})^*$ (cf. [23, Chap. 10]), where T^* denotes the adjoint, i.e., $(Tu, v)_{\dot{H}^{1/2}} = (u, T^*v)_{\dot{H}^{1/2}}$. Using this, it is straightforward to verify that with $T = \mathcal{G}^{-1}$ we get that $(\mathcal{G}^*)^{-1} = \mathcal{G}^{-1}$ and so $\mathcal{G}^* = \mathcal{G}$.

The boundary value problem associated with $\mathcal{L}$ (and η) is

$$\begin{cases} \mathcal{L}u = f & \text{in } \Gamma_S, \\ \partial_\nu u = 0 & \text{on } \partial\Gamma_S. \end{cases}$$

It has a unique solution $u \in \dot{H}^1$ for every $f \in \dot{L}^2$ (cf. [33, Chap. 8]), and u satisfies

$$S(\nabla u, \nabla v)_{L^2} + g(u, v)_{L^2} = -(f, v)_{L^2} \quad \text{for all } v \in \dot{H}^1.$$

We also have that

$$\mathcal{L}\psi_{m,n} = -(g + Sk_{m,n}^2)\psi_{m,n}, \quad m, n \in \mathbb{N}_0,$$

and so

$$u = \mathcal{L}^{-1}f = \sum_{m+n>0} \frac{\hat{f}_{m,n}}{-(g + Sk_{m,n}^2)} \psi_{m,n}.$$

By the same arguments used for $\mathcal{G}$, we can therefore conclude that $\mathcal{L} : \dot{H}^3 \rightarrow \dot{H}^1$ is a bijection and that $\mathcal{L}$ is self-adjoint on $\dot{H}^1$.

We now have what we need to prove well-posedness for the forward problem.

Proof of Theorem 3.1. In section 3, we had given the water wave system in the form

$$\dot{U} = AU, \quad U(0) = U_0, \quad \text{with} \quad A = \begin{bmatrix} 0 & \mathcal{G} \\ \mathcal{L} & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \eta \\ \varphi \end{bmatrix}.$$

We first show that $A : D(A) \rightarrow H_E$, where $D(A) = \dot{H}^{5/2} \times \dot{H}^2$ is the domain of A , is skew-adjoint. A is skew-symmetric, since

$$\langle AU, V \rangle_E = -\langle U, AV \rangle_E \quad \text{for all } U, V \in D(A).$$

Since both $\mathcal{L}$ and $\mathcal{G}$ are bijections, we have $R(A) = H_E$, where $R(A)$ denotes the range of A . We can therefore conclude that A is skew-adjoint (cf. [39, sect. 3.7]). By Stone's theorem [39, Thm. 3.8.6], A is the generator of a strongly continuous, unitary semigroup $\mathbb{T}_t$. Hence the unique solution to the initial value problem $\dot{U} = AU, U(0) = U_0$ in (3.2) is given by $U(t) = \mathbb{T}_t U_0$, and $U \in C([0, \infty); H_E)$ satisfies $\|U(t)\|_E = \|U_0\|_E$.

Moreover, A is diagonalizable. Indeed, one can check that

$$AV_{m,n}^\pm = \pm i\omega_{m,n} V_{m,n}^\pm,$$

where

$$V_{m,n}^\pm = \begin{bmatrix} (2(g + Sk_{m,n}^2))^{-1/2} \\ \pm i(2k_{m,n} \tanh(k_{m,n}H))^{-1/2} \end{bmatrix} \psi_{m,n}$$

and

$$\omega_{m,n} = \sqrt{(gk_{m,n} + Sk_{m,n}^3) \tanh(k_{m,n}H)}.$$

The eigenvectors here are chosen such that $\{V_{m,n}^+\}$ and $\{V_{m,n}^-\}$ are biorthogonal sequences, i.e., $\langle V_{m,n}^+, V_{i,j}^- \rangle_E = \delta_{m,n,i,j}$. Consequently,

$$U(t) = \sum_{m,n \geq 0} e^{i\omega_{m,n}t} \langle U_0, V_{m,n}^- \rangle_E V_{m,n}^+ + e^{-i\omega_{m,n}t} \langle U_0, V_{m,n}^+ \rangle_E V_{m,n}^-.$$

Since

$$e^{i\omega_{m,n}t} \langle U_0, V_{m,n}^- \rangle_E V_{m,n}^+ + e^{-i\omega_{m,n}t} \langle U_0, V_{m,n}^+ \rangle_E V_{m,n}^- = 2\operatorname{Re}(e^{i\omega_{m,n}t} \langle U_0, V_{m,n}^- \rangle_E V_{m,n}^+)$$

and

$$\langle U_0, V_{m,n}^- \rangle_E V_{m,n}^+ = \frac{1}{2} \begin{bmatrix} (\eta_0, \psi_{m,n})_{L^2} - i \frac{(k_{m,n} \tanh(k_{m,n}H))^{1/2}}{(g + Sk_{m,n}^2)^{1/2}} (\varphi_0, \psi_{m,n})_{L^2} \\ (\varphi_0, \psi_{m,n})_{L^2} + i \frac{(g + Sk_{m,n}^2)^{1/2}}{(k_{m,n} \tanh(k_{m,n}H))^{1/2}} (\eta_0, \psi_{m,n})_{L^2} \end{bmatrix} \psi_{m,n},$$

it follows that

$$U(t) = \sum_{m,n \geq 0} \cos(\omega_{m,n}t) \psi_{m,n} \begin{bmatrix} (\eta_0, \psi_{m,n})_{L^2} \\ (\varphi_0, \psi_{m,n})_{L^2} \end{bmatrix} + \sin(\omega_{m,n}t) \psi_{m,n} \begin{bmatrix} \frac{\omega_{m,n}}{g + Sk_{m,n}^2} (\varphi_0, \psi_{m,n})_{L^2} \\ -\frac{g + Sk_{m,n}^2}{\omega_{m,n}} (\eta_0, \psi_{m,n})_{L^2} \end{bmatrix}.$$

Last, note that for $m + n > 0$, we can find positive constants such that

$$c_1 |m^2 + n^2|^{-1/4} \leq \frac{\omega_{m,n}}{g + Sk_{m,n}^2} \leq c_2 |m^2 + n^2|^{-1/4},$$

$$\tilde{c}_1 |m^2 + n^2|^{1/4} \leq \frac{g + Sk_{m,n}^2}{\omega_{m,n}} \leq \tilde{c}_2 |m^2 + n^2|^{1/4}.$$

Assuming $(\eta_0, \varphi_0) \in \dot{H}^{s+\frac{1}{2}} \times \dot{H}^s$, we see that

$$\begin{aligned} \|\eta(X, t)\|_{\dot{H}^{s+\frac{1}{2}}} &\leq \|\eta_0\|_{\dot{H}^{s+\frac{1}{2}}} + \left\| \sum_{m,n} \frac{\omega_{m,n}}{g + Sk_{m,n}^2} (\varphi_0, \psi_{m,n})_{L^2} \psi_{m,n} \right\|_{\dot{H}^{s+\frac{1}{2}}} \\ &\leq C \left(\|\eta_0\|_{\dot{H}^{s+\frac{1}{2}}} + \|\varphi_0\|_{\dot{H}^s} \right), \\ \|\varphi(X, t)\|_{\dot{H}^s} &\leq \|\varphi_0\|_{\dot{H}^s} + \left\| \sum_{m,n} \frac{g + Sk_{m,n}^2}{\omega_{m,n}} (\eta_0, \psi_{m,n})_{L^2} \psi_{m,n} \right\|_{\dot{H}^s} \\ &\leq \tilde{C} \left(\|\varphi_0\|_{\dot{H}^s} + \|\eta_0\|_{\dot{H}^{s+\frac{1}{2}}} \right). \end{aligned} \quad \square$$

7.2. Proofs of Theorem 4.4 and Corollary 4.6.

Measurements and Nonharmonic Fourier Series. The measurement operator takes the form

$$(7.3) \quad \mathcal{M}(X, t) = \sum_{m,n \geq 0}^{\infty} \psi_{m,n}(X) \left(\cos(\omega_{m,n}t) q_{m,n} + \sin(\omega_{m,n}t) \frac{\omega_{m,n}}{g + Sk_{m,n}^2} p_{m,n} \right),$$

with $q_{m,n} = (\eta_0, \psi_{m,n})_{L^2(\Gamma_S)}$ and $p_{m,n} = (\varphi_0, \psi_{m,n})_{L^2(\Gamma_S)}$. Setting

$$c_{m,n} = \frac{1}{2} \left(q_{m,n} - i \frac{\omega_{m,n}}{g + Sk_{m,n}^2} p_{m,n} \right) \quad \text{for } m, n \geq 0,$$

we may write

$$\mathcal{M}(X, t) = \sum_{m \geq 0} \left(\sum_{n \geq 0}^{\infty} \psi_{m,n}(X) c_{m,n} e^{i\omega_{m,n}t} + c.c. \right),$$

where *c.c.* denotes the complex conjugate of $\sum_{n \geq 0}^{\infty} \psi_{m,n}(X) c_{m,n} e^{i\omega_{m,n}t}$. Fixing m and temporarily ignoring the spatial dependence in the above expression, we have a function in the form

$$(7.4) \quad f_m(t) = \sum_{n \geq 0} c_{m,n} e^{i\omega_{m,n}t} + c.c.$$

This is an example of a nonharmonic Fourier series (or almost periodic functions) [41, 39]. They are functions in the form

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{i\lambda_n t}, \quad (a_n) \in \ell^2, \lambda_n \in \Lambda,$$

where the frequency set $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ differs from the classical $\{2\pi n/T\}_{n \in \mathbb{Z}}$. When the frequency set is nonharmonic, as is the case with $\{\omega_{m,n}\}$, we cannot immediately apply Fourier inversion. But under certain conditions, we can get a result similar to Parseval's identity: Ingham's theorem (cf. [39]) states that if $\inf_{m \neq n} |\lambda_n - \lambda_m| = \gamma > 0$ and $T > 2\pi/\gamma$, then there are positive constants c_-, c_+ such that

$$c_- \sum_n |a_n|^2 \leq \int_0^T \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \leq c_+ \sum_n |a_n|^2.$$

For fixed m (or n), the set $\{\omega_{m,n}\}_n$ satisfies the requirements of Ingham's theorem, and we use this in subsection 4.3. However, it is easy to check that $|\omega_{m,0} - \omega_{m,1}| \rightarrow 0$ as $m \rightarrow \infty$; i.e., if both indices are allowed to vary, the gap condition does not hold. To circumvent this, we will use a weaker version of Ingham's theorem from [26] that was used to prove observability of the wave equation in [27, 22].

THEOREM 7.1 (weakened Ingham's theorem [22, Prop. 3.1]). *Let $\{\lambda_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers and N a positive integer, and assume there exists some $\gamma > 0$ such that*

$$|\lambda_k - \lambda_l| \geq |k - l|\gamma \quad \text{if} \quad \max(|k|, |l|) \geq N.$$

Then for any $(a_n) \in \ell^2$ and $T > 0$,

$$(7.5) \quad \int_0^T \left| \sum_{k \in \mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt \geq \frac{2T}{\pi} \left(\sum_{|k| \geq N} |a_k|^2 - \left(\frac{2\pi}{T\gamma} \right)^2 \sum_{k \in \mathbb{Z}} |a_k|^2 \right).$$

We now adapt the observability proof from [27, 22] to water waves. To show that Theorem 7.1 is satisfied for $f_m(t)$ in (7.4), we note that with

$$\lambda_k^m = \begin{cases} \operatorname{sgn}(k)\omega_{m,|k|-1}, & |k| > 0, \\ 0, & k = 0, \end{cases} \quad \text{and} \quad a_k^m = \begin{cases} c_{m,k-1}, & k > 0 \\ \bar{c}_{m,|k|-1}, & k < 0, \\ 0, & k = 0, \end{cases} \quad \text{for } k \in \mathbb{Z},$$

we have

$$\sum_{k \in \mathbb{Z}} a_k^m e^{i\lambda_k^m t} = f_m(t).$$

Hence, if the gap condition in Theorem 7.1 holds for the frequency set $\{\lambda_k^m\}_{k \in \mathbb{Z}}$, then (7.5) holds for $f_m(t)$.

LEMMA 7.2. *For any $m, n \in \mathbb{N}_0$, let*

$$\{\lambda_k^m = \operatorname{sgn}(k)\omega_{m,|k|-1}\}_{k \in \mathbb{Z}} \quad \text{and} \quad \{\tilde{\lambda}_k^n = \operatorname{sgn}(k)\omega_{|k|-1,n}\}_{k \in \mathbb{Z}}.$$

It then holds for $k, l \in \mathbb{Z}$ that

$$\begin{aligned} |\lambda_k^m - \lambda_l^m| &\geq |k - l|\gamma \quad \text{for } \max(|k|, |l|) \geq m, \\ |\tilde{\lambda}_k^n - \tilde{\lambda}_l^n| &\geq |k - l|\gamma \quad \text{for } \max(|k|, |l|) \geq n, \end{aligned}$$

with

$$\gamma = \frac{\pi W}{2L\sqrt{L^2 + W^2}} \sqrt{\frac{Hg}{2\sqrt{1 + \frac{H^2 g}{3S}}} - \frac{3S}{2H}}.$$

Proof. If $\operatorname{sgn}(k) = \operatorname{sgn}(l)$, then $|\lambda_k^m - \lambda_l^m| = |\omega_{m,|k|-1} - \omega_{m,|l|-1}|$. On the other hand, for $k, l > 0$,

$$\lambda_k^m - \lambda_l^m = \omega_{m,k-1} + \omega_{m,l-1} \geq \omega_{m,k-1} - \omega_{m,l-1},$$

and so if for any $p, q \geq 0$ it holds that

$$|\omega_{m,p} - \omega_{m,q}| \geq |p - q|\gamma \quad \text{and} \quad \omega_{m,p} \geq p\gamma \quad \text{for } \max(p, q) \geq m,$$

then the gap condition holds for $\{\lambda_k\}_{k \in \mathbb{Z} \setminus \{0\}}$.

We now consider the function $G(t) = \sqrt{(gt + St^3) \tanh(tH)}$ for $t > 0$. We have

$$G'(t) = \frac{(g + 3St^2) \tanh(tH) + H(gt + St^3) \operatorname{sech}(tH)^2}{2\sqrt{(gt + St^3) \tanh(tH)}}.$$

Using elementary inequalities together with the lower bound $\tanh(t) > t/\sqrt{1+t^2}$, $t > 0$ (see [6, Lem. 4]), we find that

$$G'(t) > \sqrt{\frac{H(g + 3St^2)}{4(1 + Ht)}} \geq \sqrt{\frac{Hg}{2\sqrt{1 + \frac{H^2g}{3S}}} - \frac{3S}{2H}} = \tilde{\gamma}.$$

Fixing m and assuming $p \geq m$ and $p > q \geq 0$, we have that

$$k_{m,p} \leq \pi p \sqrt{L^{-2} + W^{-2}} < \pi(p+q) \sqrt{L^{-2} + W^{-2}} \quad \text{and} \quad k_{m,p}^2 - k_{m,q}^2 = \frac{\pi^2}{W^2}(p^2 - q^2).$$

Hence

$$|k_{m,p} - k_{m,q}| = \frac{|k_{m,p}^2 - k_{m,q}^2|}{|k_{m,p} + k_{m,q}|} > \frac{\frac{\pi^2}{W^2}|p - q||p + q|}{2\pi|p + q|\sqrt{L^{-2} + W^{-2}}} \geq \frac{\pi L}{2W\sqrt{L^2 + W^2}}|p - q|.$$

We obtain

$$(7.6) \quad \begin{aligned} |\omega_{m,p} - \omega_{m,q}| &= |G(k_{m,p}) - G(k_{m,q})| \\ &= \left| \int_{k_{m,p}}^{k_{m,q}} G'(t) dt \right| \geq \frac{\pi L}{2W\sqrt{L^2 + W^2}}|p - q|\tilde{\gamma}. \end{aligned}$$

We also have that $k_{m,p} \geq \frac{\pi}{W}p$ and so $\omega_{m,p} \geq \frac{\pi}{W}p\tilde{\gamma}$. Arguing the same way, we find the estimate

$$|\omega_{p,n} - \omega_{q,n}| \geq \frac{\pi W}{2L\sqrt{L^2 + W^2}}|p - q|\tilde{\gamma},$$

and since we have assumed $L \geq W$, the result follows. $\square$

We now prove Theorem 4.4.

Proof of Theorem 4.4. On the boundary Γ_M , we have

$$\begin{aligned} \mathcal{M}(X, t)|_{x_2=0} &= \sum_{m \geq 0} \frac{C_{m,|n|}}{\sqrt{WL}} \cos(\pi m x_1/L) \sum_{n \in \mathbb{Z}} a_k^n e^{i\lambda_n^m t}, \\ \mathcal{M}(X, t)|_{x_1=0} &= \sum_{n \geq 0} \frac{C_{|m|,n}}{\sqrt{WL}} \cos(\pi n x_2/L) \sum_{m \in \mathbb{Z}} b_m^n e^{i\tilde{\lambda}_m^n t}, \end{aligned}$$

where

$$b_m^n = \begin{cases} c_{m-1,n}, & m > 0, \\ \bar{c}_{|m|-1,n}, & km0, \\ 0, & m = 0, \end{cases} \quad \text{for } k \in \mathbb{Z}.$$

We write

$$I_1 = \int_0^T \int_0^L |\mathcal{M}(X, t)|^2 dx_1 dt \quad \text{and} \quad I_2 = \int_0^T \int_0^W |\mathcal{M}(X, t)|^2 dx_2 dt.$$

By orthogonality, Theorem 7.1, and Lemma 7.2, we have

$$I_1 = \frac{2}{W} \sum_{n \geq 0} \int_0^T \left| \sum_{n \in \mathbb{Z}} a_n^m e^{i\lambda_n^m t} \right|^2 dt \geq \frac{4T}{W\pi} \sum_{m \geq 0} \left(\sum_{\substack{n \in \mathbb{Z} \\ |n| \geq m}} |a_n^m|^2 - \left(\frac{2\pi}{T\gamma} \right)^2 \sum_{n \in \mathbb{Z}} |a_n^m|^2 \right)$$

and

$$I_2 = \frac{2}{L} \sum_{n \geq 0} \int_0^T \left| \sum_{m \in \mathbb{Z}} b_m^n e^{i\tilde{\lambda}_m^n t} \right|^2 dt \geq \frac{4T}{L\pi} \sum_{n \geq 0} \left(\sum_{\substack{m \in \mathbb{Z} \\ |m| \geq n}} |b_m^n|^2 - \left(\frac{2\pi}{T\gamma} \right)^2 \sum_{m \in \mathbb{Z}} |b_m^n|^2 \right).$$

Consequently,

$$I_1/L + I_2/W \geq \frac{4T}{WL\pi} \left(\sum_{m \geq 0} \sum_{\substack{n \in \mathbb{Z} \\ |n| \geq m}} |a_n^m|^2 + \sum_{n \geq 0} \sum_{\substack{m \in \mathbb{Z} \\ |m| \geq n}} |b_m^n|^2 - \left(\frac{2\pi}{T\gamma} \right)^2 \sum_{k \geq 0} \sum_{l \in \mathbb{Z}} |a_l^k|^2 + |b_l^k|^2 \right).$$

In terms of $c_{m,n}$, we have that

$$\begin{aligned} \sum_{m \geq 0} \sum_{\substack{n \in \mathbb{Z} \\ |n| \geq m}} |a_n^m|^2 + \sum_{n \geq 0} \sum_{\substack{m \in \mathbb{Z} \\ |m| \geq n}} |b_m^n|^2 &= 2 \sum_{m \geq 0} \sum_{\substack{n \in \mathbb{N} \\ n \geq m}} |c_{m,n-1}|^2 + 2 \sum_{n \geq 0} \sum_{\substack{m \in \mathbb{N} \\ m \geq n}} |c_{m-1,n}|^2 \\ &\geq 2 \sum_{m \geq 0} \sum_{n \geq 0} |c_{m,n}|^2 \end{aligned}$$

and

$$\sum_{k \geq 0} \sum_{l \in \mathbb{Z}} |a_l^k|^2 + |b_l^k|^2 = 2 \sum_{m \geq 0} \sum_{n \geq 0} |c_{m,n}|^2.$$

As a result,

$$\begin{aligned} I_1/L + I_2/W &\geq \frac{8T}{WL\pi} \left(\left(1 - \left(\frac{2\pi}{T\gamma} \right)^2 \right) \sum_{m \geq 0} \sum_{n \geq 0} |c_{m,n}|^2 \right) \\ &= \frac{8T}{WL\pi} \left(\left(1 - \left(\frac{2\pi}{T\gamma} \right)^2 \right) \sum_{m \geq 0} \sum_{n \geq 0} q_{m,n}^2 + \left(\frac{\omega_{m,n}}{g + Sk_{m,n}^2} p_{m,n} \right)^2 \right). \end{aligned}$$

Since $\frac{\omega_{m,n}}{g + Sk_{m,n}^2} \sim (m^2 + n^2)^{-1/4}$, there is a constant C_s such that

$$\sum_{m \geq 0} \sum_{n \geq 0} q_{m,n}^2 + \left(\frac{\omega_{m,n}}{g + Sk_{m,n}^2} p_{m,n} \right)^2 \geq C_s (\|\eta_0\|_{L^2}^2 + \|\varphi_0\|_{\dot{H}^{-1/2}}^2).$$

With T as in Theorem 4.4 we get $1 - \left(\frac{2\pi}{T\gamma} \right)^2 = 3/4$. Then, with $C = \frac{2}{L\pi} C_s$, we have

$$\int_0^T \int_{\Gamma_M} |\mathcal{M}(X, t)|^2 dX dt = I_1 + I_2 \geq TC (\|\eta_0\|_{L^2}^2 + \|\varphi_0\|_{\dot{H}^{-1/2}}^2). \quad \square$$

Proof of Corollary 4.6. For $k = 1, 2, \dots$ we have that $\omega_{m,n}^{2k} \sim (m^2 + n^2)^{3k/2}$. Assuming that the initial data is sufficiently smooth, we differentiate $\mathcal{M}$ k times with respect to t and carry out the argument above to find

$$I_1/L + I_2/W = \frac{8T}{WL\pi} \left(\left(1 - \left(\frac{2\pi}{T\gamma} \right)^2 \right) \sum_{m \geq 0} \sum_{n \geq 0} q_{m,n}^2 \omega_{m,n}^{2k} + \left(\frac{\omega_{m,n}^{1+k}}{g + Sk_{m,n}^2} p_{m,n} \right)^2 \right).$$

The conclusion now follows since there is a constant C_k such that

$$\sum_{m \geq 0} \sum_{n \geq 0} q_{m,n}^2 \omega_{m,n}^{2k} + \left(\frac{\omega_{m,n}^{1+k}}{g + Sk_{m,n}^2} p_{m,n} \right)^2 \geq C_k \left(\|\eta_0\|_{\dot{H}^{\frac{3}{2}k}}^2 + \|\varphi_0\|_{\dot{H}^{\frac{3}{2}k-1/2}}^2 \right). \quad \square$$

7.3. Proof of Theorem 4.7. The idea of this proof is to construct a well-conditioned mapping from the discrete measurement to the Fourier coefficients of the initial disturbance. It is achieved by showing that a finite-dimensional version of Ingham's theorem holds under the right conditions.

Proof. Set $\mathcal{A} = [a_n^m]$ and $\mathcal{C} = [C_{m,|n|} a_n^m]$ as the matrix of the Fourier coefficients of $P_\beta \eta_0$, where $C_{m,|n|}$ are the normalization constants for $\psi_{m,n}$, and $\|P_\beta \eta_0\|_{L^2} = \|\mathcal{A}\|_F$. We have that

$$P_\beta \eta(X, t)|_{\Gamma_M} = \sum_{0 \leq m \leq \beta} \cos(\pi m x_1 / L) \sum_{0 < |n| \leq \beta+1} C_{m,|n|} a_n^m e^{i\lambda_n^m t},$$

and for a fixed time t_j , we get

$$\mathcal{M}_\beta(k, j) = \sum_{0 \leq m \leq \beta} \cos(\pi m(k + 1/2)/N_x) \sum_{0 < |n| \leq \beta+1} C_{m,|n|} a_n^m e^{i\lambda_n^m t_j}, \quad 0 \leq k \leq N_x - 1.$$

Writing

$$u_j = \begin{bmatrix} \sum_{0 < |n| \leq \beta+1} a_n^0 e^{i\lambda_n^0 t_j} \\ \sum_{0 < |n| \leq \beta+1} a_n^1 e^{i\lambda_n^1 t_j} \\ \vdots \\ \sum_{0 < |n| \leq \beta+1} a_n^\beta e^{i\lambda_n^\beta t_j} \end{bmatrix},$$

we see that the measurement at time t_j is given by² $\mathcal{M}_\beta(:, j) = D_3 u_j$, where D_3 is the discrete cosine transform matrix of the third type (cf. [37]), i.e.,

$$D_3 = [\cos(\pi(k + 1/2)m/N_x)] \in \mathbb{R}^{N_x \times N_x}, \quad 0 \leq k \leq N_x - 1, \quad 0 \leq m \leq N_x - 1.$$

The matrix D_3 is orthogonal and can be made unitary by scaling the first row by $1/\sqrt{N}$ and the others by $\sqrt{2/N}$. Writing $U = [u_0 \ u_1 \ \dots \ u_{N_t}]$, we have

$$\mathcal{M}_\beta = D_3 U \implies U = D_3^{-1} \mathcal{M}_\beta.$$

The point of this is that row $U(m, :)$ contains the sampled values of $\sum_{0 < |n| \leq \beta+1} a_n^m e^{i\lambda_n^m t}$. Recalling that $\lambda_n^m = \text{sgn}(n)\omega_{m,|n|-1}$, we define the nonharmonic Vandermonde matrix

²We use the notation $V(:, k)$ and $V(n, :)$ to denote the k th column and n th row of the matrix V , respectively.

V_m as follows:³ Let $v_n^m = e^{i\omega_{m,n}\Delta t}$ and

$$(7.7) \quad V_m(j, :) = [(\overline{v_\beta^m})^j \quad \cdots \quad (\overline{v_0^m})^j \quad (v_0^m)^j \quad \cdots \quad (v_\beta^m)^j] \quad \text{for } 0 \leq j \leq N_t.$$

In terms of V_m , we can write

$$U(m, :)^{\top} = V_m C(m, :)^{\top}.$$

To show that V_m is well-conditioned with the choice of T and N_t , we rely on a type of discrete Ingham's inequality (cf. [5] or [29, Thm. 10.23]). We set $\{\xi_n\}_{n \leq N_\beta} \subset [-1/2, 1/2)$ and let

$$(7.8) \quad \tilde{\delta} = \min_{k \neq l} \left(\min_{n \in \mathbb{Z}} |\xi_k - \xi_l - n| \right).$$

Let $v_n = \exp(i2\pi\xi_n)$, and let $N_t > \max(N_\beta, 1/\tilde{\delta})$. Then the Vandermonde matrix

$$V = [v_n^j], \quad 0 \leq j \leq N_t, \quad 0 \leq n \leq N_\beta,$$

satisfies

$$(N_t - 1/\tilde{\delta})\|x\|_2^2 \leq \|Vx\|_2^2 \leq (N_t + 1/\tilde{\delta})\|x\|_2^2 \quad \text{for all } x \in \mathbb{C}^{N_\beta}.$$

We now show that under the assumptions on N_t and T in Theorem 4.7, the matrix V_m defined in (7.7) satisfies the conditions of the discrete Ingham's inequality. For fixed $m \geq 0$, we have

$$\{\xi_n\} = \{-\omega_{m,n}\Delta t/2\pi\}_{n \leq \beta} \cup \{\omega_{m,n}\Delta t/2\pi\}_{n \leq \beta}.$$

The so-called wrap-around metric in equation (7.8) ensures that the matrix entries $e^{i2\pi\xi_n}$ are separated. For example, for $\xi_0 = 0$, $\xi_1 = 0.75$, and $\xi_2 = 1$, we have $\min_{n \in \mathbb{Z}} |\xi_0 - \xi_2 - n| = 0$ and $\min_{n \in \mathbb{Z}} |\xi_0 - \xi_1 - n| = 0.25$. We choose $\Delta t = \frac{4\pi}{5\omega_{\beta,\beta}}$, as this ensures that $\{\omega_{m,n}\Delta t/2\pi\}_{|n| \leq \beta} \subset [-2/5, 2/5]$ for all $m \leq \beta$. From Lemma 7.2 we find a bound on the distance between $\omega_{m,n}$ for varying n and fixed m : Assume $n, p \geq 0$. From (7.6) in Lemma 7.2, we have that $|\omega_{m,n} - \omega_{m,p}| \geq |k_{m,n} - k_{m,p}|\tilde{\gamma}$. For $m > 0$,

$$\begin{aligned} \min_{n \neq p} |k_{m,n} - k_{m,p}| &= |k_{m,1} - k_{m,0}| = \pi \frac{m}{L} \left(\sqrt{1 + (L/Wm)^2} - 1 \right) \\ &\geq \pi \frac{\beta}{L} \left(\sqrt{1 + (L/W\beta)^2} - 1 \right) \geq \frac{\pi}{L} \left(\sqrt{\beta^2 + 1} - \beta \right). \end{aligned}$$

Since $\pi/W \geq \frac{\pi}{L}(\sqrt{\beta^2 + 1} - \beta)$, this estimate also holds for $m = 0$. Finally, the smallest distance between two frequencies with opposite signs satisfies $2\omega_{1,0} \geq k_{1,0}\tilde{\gamma}$, and we get a lower bound on the minimal wrap-around distance:

$$\delta = \min_{k \neq l} \left(\min_{n \in \mathbb{Z}} \left| \omega_{m,k} \frac{\Delta t}{2\pi} - \omega_{m,l} \frac{\Delta t}{2\pi} - n \right| \right) \geq \frac{\Delta t}{2\pi} \min \left(1/5, \frac{\pi}{L} \left(\sqrt{\beta^2 + 1} - \beta \right) \tilde{\gamma} \right).$$

³The indexing is awkward, since $\omega_{0,0} = 0$ is included. But since $p_{0,0} = (\eta_0, \varphi_{0,0})_{L^2} = 0$, we can just remove the columns with $\omega_{0,0}$ -terms from V_0 . We will assume this and not mention it in what follows.

Recalling the choice of $N_t = \lceil 1 + \frac{1}{\delta} \rceil$ and $T = \frac{4\pi N_t}{5\omega_{\beta,\beta}}$, we have $\Delta t = \frac{T}{N_t} = \frac{4\pi}{5\omega_{\beta,\beta}}$. Hence, with $N_\beta = 2\beta + 2$, the conditions of discrete Ingham's inequality hold. For each m , we therefore have

$$\begin{aligned} \left(N_t - \frac{1}{\delta}\right) \|\mathcal{C}(m, :)\|_2^2 &\leq \|V_m \mathcal{C}(m, :)\|_2^2 = \|U(m, :)\|_2^2 \\ &= \|(D_3^{-1} \mathcal{M})^\top(m, :)\|_2^2 \leq \|D_3^{-1}\|_2^2 \|\mathcal{M}(:, m)\|_2^2. \end{aligned}$$

Taking $S = \text{diag}(1/\sqrt{N_x}, \sqrt{2/N_x}, \dots, \sqrt{2/N_x})$, we have that SD_3 is unitary, and so $D_3^{-1} = D_3^\top S^2$. Consequently, we find that

$$\|D_3^{-1}\|_2 = \sqrt{\lambda_{\max}(D_3^{-1}(D_3^{-1})^\top)} = \sqrt{\lambda_{\max}(D_3^\top S(S^2)SD_3)} = \sqrt{\frac{2}{N_x}}.$$

Hence

$$\|\mathcal{C}\|_F^2 = \sum_m \|\mathcal{C}(m, :)\|_2^2 \leq \sum_m \frac{2/N_x}{(N_t - \frac{1}{\delta})} \|\mathcal{M}_\beta(:, m)\|_2^2 \leq \frac{2}{N_x(N_t - \frac{1}{\delta})} \|\mathcal{M}_\beta\|_F^2,$$

and since $\|\mathcal{A}\|_F^2 \leq \frac{4}{LW} \|\mathcal{C}\|_F^2$, we get

$$\|\mathcal{A}\|_F^2 \leq \frac{LW}{2N_x(N_t - \frac{1}{\delta})} \|\mathcal{M}_\beta\|_F^2.$$

Denote by $\mathcal{A}_R$ the matrix of Fourier coefficients of the reconstruction $R_\beta \eta_0$ obtained by applying the procedure outlined above⁴ to $\mathcal{M}$. It now follows by linearity that

$$\|P_\beta \eta_0 - R_\beta \eta_0\|_{L^2}^2 = \|\mathcal{A} - \mathcal{A}_R\|_F^2 \leq \frac{LW}{2N_x(N_t - \frac{1}{\delta})} \|\mathcal{M}_\beta - \mathcal{M}\|_F^2.$$

Last, for $f \in \dot{H}^s$, $s > 0$, we have that

$$\|f - P_\beta f\|_{L^2}^2 = \sum_{m,n>\beta} \hat{f}_{m,n}^2 \leq 2\beta^{-2s} \sum_{m,n>\beta} (m^2 + n^2)^s \hat{f}_{m,n}^2,$$

and hence

$$\|f - P_\beta f\|_{L^2} \leq \frac{\sqrt{2}}{\beta^s} \|f\|_{\dot{H}^s}.$$

Consequently, we get

$$\begin{aligned} \|\eta_0 - R_\beta \eta_0\|_{L^2} &= \|\eta_0 - P_\beta \eta_0 + P_\beta \eta_0 - R_\beta \eta_0\|_{L^2} \leq \|\eta_0 - P_\beta \eta_0\|_{L^2} + \|P_\beta \eta_0 - R_\beta \eta_0\|_{L^2} \\ &\leq \frac{\sqrt{2}}{\beta^s} \|f\|_{\dot{H}^s} + \sqrt{\frac{LW}{2N_x(N_t - \frac{1}{\delta})}} \|\mathcal{M}_\beta - \mathcal{M}\|_F. \quad \square \end{aligned}$$

REFERENCES

- [1] M. J. ABLOWITZ, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons*, Cambridge University Press, 2011. (Cited on pp. 696, 697, 702, 706)

⁴See subsection 5.2.

- [2] T. ALAZARD, P. BALDI, AND D. HAN-KWAN, *Control of water waves*, J. European Math. Soc., 20 (2018), pp. 657–745. (Cited on p. 695)
- [3] G. S. ALBERTI AND Y. CAPDEBOSCQ, *Lectures on Elliptic Methods for Hybrid Inverse Problems*, Cours Spéc. 25, Société Mathématique de France, Paris, 2018. (Cited on p. 702)
- [4] G. S. ALBERTI AND M. SANTACESARIA, *Infinite-dimensional inverse problems with finite measurements*, Arch. Ration. Mech. Anal., 243 (2022), pp. 1–31. (Cited on p. 702)
- [5] C. AUBEL AND H. BÖLCSKEI, *Vandermonde matrices with nodes in the unit disk and the large sieve*, Appl. Comput. Harmon. Anal., 47 (2019), pp. 53–86. (Cited on p. 715)
- [6] Y. J. BAGUL, R. M. DHAIGUDE, C. CHESNEAU, AND M. KOSTIĆ, *Tight exponential bounds for hyperbolic tangent*, Jordan J. Math. Statist., 15 (2022), pp. 807–821. (Cited on p. 712)
- [7] C. BARDOS, G. LEBEAU, AND J. RAUCH, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065, <https://doi.org/10.1137/0330055>. (Cited on p. 701)
- [8] T. J. BRIDGES, M. D. GROVES, AND D. P. NICHOLLS, *Lectures on the Theory of Water Waves*, London Math. Soc. Lecture Note Ser. 426, Cambridge University Press, 2016. (Cited on p. 697)
- [9] M. CHOULLI, *Applications of Elliptic Carleman Inequalities to Cauchy and Inverse Problems*, Springer, 2016. (Cited on p. 700)
- [10] G. COVI, M. DE HOOP, AND M. SALO, *Uniqueness in an inverse problem of fractional elasticity*, Proc. A, 479 (2023), art. 20230474. (Cited on p. 700)
- [11] W. CRAIG, *Surface water waves and tsunamis*, J. Dynam. Differential Equations, 18 (2006), pp. 525–549. (Cited on p. 697)
- [12] W. CRAIG, *A Course on Partial Differential Equations*, Grad. Stud. Math. 197, American Mathematical Society, 2018. (Cited on p. 700)
- [13] A. ELCRAT, V. ISAKOV, E. KROPP, AND D. STEWART, *A stability analysis of the harmonic continuation*, Inverse Problems, 28 (2012), art. 075016. (Cited on p. 701)
- [14] M. A. FONTELOS, R. LECAROS, J. C. LÓPEZ-RÍOS, AND J. H. ORTEGA, *Bottom detection through surface measurements on water waves*, SIAM J. Control Optim., 55 (2017), pp. 3890–3907, <https://doi.org/10.1137/15M1007951>. (Cited on p. 695)
- [15] T. GHOSH, A. RÜLAND, M. SALO, AND G. UHLMANN, *Uniqueness and reconstruction for the fractional Calderón problem with a single measurement*, J. Funct. Anal., 279 (2020), art. 108505. (Cited on p. 700)
- [16] T. GHOSH, M. SALO, AND G. UHLMANN, *The Calderón problem for the fractional Schrödinger equation*, Anal. PDE, 13 (2020), pp. 455–475. (Cited on p. 700)
- [17] B. HARRACH, *Uniqueness, stability and global convergence for a discrete inverse elliptic Robin transmission problem*, Numer. Math., 147 (2021), pp. 29–70. (Cited on p. 702)
- [18] R. S. JOHNSON, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, 1997. (Cited on p. 697)
- [19] B. KALTENBACHER AND W. RUNDELL, *Inverse Problems for Fractional Partial Differential Equations*, Grad. Stud. Math. 230, American Mathematical Society, 2023. (Cited on p. 700)
- [20] C. E. KENIG, D. PILOD, G. PONCE, AND L. VEGA, *On the unique continuation of solutions to non-local non-linear dispersive equations*, Comm. Partial Differential Equations, 45 (2020), pp. 872–886. (Cited on p. 700)
- [21] C. E. KENIG, G. PONCE, AND L. VEGA, *Uniqueness properties of solutions to the Benjamin-Ono equation and related models*, J. Funct. Anal., 278 (2020), art. 108396. (Cited on p. 700)
- [22] V. KOMORNIK AND B. MIARA, *Cross-like internal observability of rectangular membranes*, Evol. Equ. Control Theory, 3 (2014), pp. 135–146. (Cited on pp. 701, 711)
- [23] E. KREYSZIG, *Introductory Functional Analysis with Applications*, John Wiley & Sons, 1991. (Cited on p. 708)
- [24] N. G. KUZNETSOV, V. MAZ'YA, AND B. VAINBERG, *Linear Water Waves: A Mathematical Approach*, Cambridge University Press, 2002. (Cited on p. 697)
- [25] D. LANNES, *The Water Waves Problem: Mathematical Analysis and Asymptotics*, American Mathematical Society, 2013. (Cited on pp. 698, 706)
- [26] P. LORETI AND V. VALENTE, *Partial exact controllability for spherical membranes*, SIAM J. Control Optim., 35 (1997), pp. 641–653, <https://doi.org/10.1137/S036301299526962X>. (Cited on p. 711)
- [27] M. MEHRENBARGER, *An Ingham type proof for the boundary observability of a $N - d$ wave equation*, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 63–68. (Cited on p. 711)
- [28] J. L. MUELLER AND S. SILTANEN, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM, 2012, <https://doi.org/10.1137/1.9781611972344>. (Cited on pp. 695, 707)
- [29] G. PLONKA, D. POTTS, G. STEIDL, AND M. TASCHE, *Numerical Fourier Analysis*, Springer, 2018. (Cited on p. 715)

- [30] R. M. REID, *Control time for gravity-capillary waves on water*, SIAM J. Control Optim., 33 (1995), pp. 1577–1586, <https://doi.org/10.1137/S0363012992226351>. (Cited on pp. 695, 699)
- [31] R. M. REID AND D. L. RUSSELL, *Boundary control and stability of linear water waves*, SIAM J. Control Optim., 23 (1985), pp. 111–121, <https://doi.org/10.1137/0323009>. (Cited on p. 695)
- [32] A. RÜLAND AND M. SALO, *The fractional Calderón problem: Low regularity and stability*, Nonlinear Anal., 193 (2020), art. 111529. (Cited on p. 700)
- [33] S. SALSA, *Partial Differential Equations in Action: From Modelling to Theory*, Unitext 99, Springer, 2016. (Cited on p. 708)
- [34] O. SCHERZER, *Handbook of Mathematical Methods in Imaging*, Springer Science & Business Media, 2010. (Cited on p. 695)
- [35] M. SELLIER, *Inverse problems in free surface flows: A review*, Acta Mech., 227 (2016), pp. 913–935. (Cited on p. 695)
- [36] B. K. SMELTZER, E. ÆSØY, A. ÅDNØY, AND S. Å. ELLINGSEN, *An improved method for determining near-surface currents from wave dispersion measurements*, J. Geophys. Res. Oceans, 124 (2019), pp. 8832–8851. (Cited on p. 695)
- [37] G. STRANG, *The discrete cosine transform*, SIAM Rev., 41 (1999), pp. 135–147, <https://doi.org/10.1137/S0036144598336745>. (Cited on p. 714)
- [38] G. TENENBAUM AND M. TUCSNAK, *Fast and strongly localized observation for the Schrödinger equation*, Trans. Amer. Math. Soc., 361 (2009), pp. 951–977. (Cited on pp. 700, 701)
- [39] M. TUCSNAK AND G. WEISS, *Observation and Control for Operator Semigroups*, Springer Science & Business Media, 2009. (Cited on pp. 701, 709, 710)
- [40] V. VASAN AND B. DECONINCK, *The inverse water wave problem of bathymetry detection*, J. Fluid Mech., 714 (2013), pp. 562–590. (Cited on p. 695)
- [41] R. M. YOUNG, *An Introduction to Nonharmonic Fourier Series*, revised 1st ed., Elsevier, 2001. (Cited on p. 710)
- [42] B. ZHANG, *Unique continuation for the Korteweg–de Vries equation*, SIAM J. Math. Anal., 23 (1992), pp. 55–71, <https://doi.org/10.1137/0523004>. (Cited on p. 700)